10/7/2016 >
(44)

<u>proof</u> (roughly D'Alembert's 1746 proof)

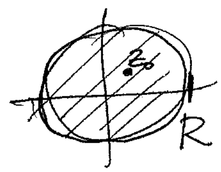
We'd like to say the root z_0 is the $z_0 \in \mathbb{C}$ achieving the <u>minimum</u> value $f(z_0)=0$ where $f(z)=|p(z)|$ i.e. $f: \mathbb{C} \xrightarrow{p} \mathbb{C} \xrightarrow{|\cdot|} \mathbb{R}$

$z \longmapsto p(z) \longmapsto |p(z)|$

Does $f: \mathbb{C} \to \mathbb{R}$ <u>achieve</u> a minimum?
Certainly, f is <u>continuous</u> (Why?)

But $\mathbb{C}$ is <u>not</u> <u>compact</u> (this is a problem for $\frac{1}{1+|z|^2}$ and for e^z)

We claim that $f(z)=|p(z)|$ should achieve a minimum value on $f(z_0)$



$|z| \le R$ for some <u>choice</u> of R, which <u>is</u> compact, (by Extreme Value Thm)

and this should be its <u>global</u> minimum $f(z_0)$

because $f(z) = |p(z)| = |z^k + a_{k-1}z^{k-1} + \ldots + a_1 z + a_0| \ge |a_0| = f(0)$

§0.7: Recall $z=a+ib$ has $|z|=\sqrt{a^2+b^2}$ and $|z_1 z_2| = |z_1||z_2|$

for R <u>sufficiently</u> <u>large</u>. How large?

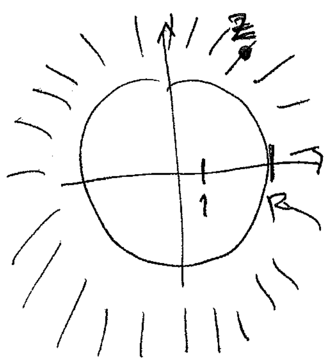
triangle inequality

If $|z|>R$, then $f(z) = |p(z)| \ge \underbrace{|z^k|}_{\ge R^k} - \underbrace{|a_{k-1}z^{k-1} + \ldots + a_1 z + a_0|}$

(should <u>dominate</u> for R large)

$\le |a_{k-1}||z^{k-1}| + \ldots + |a_1||z^1| + |a_0|$

$\le k \cdot \underbrace{\max\{|a_{k-1}|, \ldots, |a_0|\}}_{A:=} \cdot R^{k-1}$



$$
\begin{aligned}
f(z) &\ge R^k - kAR^{k-1} \\
&= R^{k-1}(R-kA) \\
&\ge R^{k-1}A && \text{if } R \ge (k+1)A \\
&\ge A && \text{if } R \ge 1 \\
&\ge |a_0| && \text{if we choose } R \ge \max\{1, (k+1)A\}
\end{aligned}
$$

Now since z_0 achieves minimum of $|p(z_0)|$, we want to show $|p(z_0)|=0$ i.e. $p(z_0)=0$. If <u>not</u>, so $|p(z_0)|>0$, we'll show $\exists$ some z_1 in a <u>small</u> circle around z_0 with $|p(z_1)| < |p(z_0)|$.

The algebra is easier if we replace $p(z)$ by $q(z) = p(z+z_0)$

(i.e. $p(z) = q(z-z_0)$)

which has same values for $|q(z)|$, so still has minimum value $|q(0)| = |p(z_0)|$, and $q(z)$ is still a degree k polynomial since

$$q(z) = p(z+z_0) = (z+z_0)^k + a_{k-1}(z+z_0)^{k-1} + \dots + a_1(z+z_0) + a_0$$

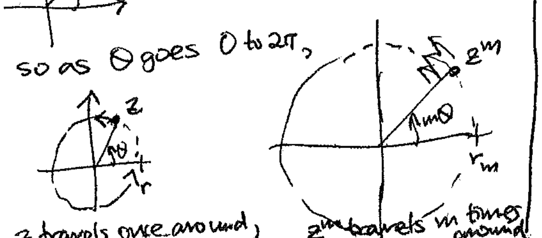
$$= z^k + b_{k-1}z^{k-1} + \dots + b_1 z^1 + b_0$$

$b_0 = q(0) = p(z_0) \neq 0$ by assumption

Now write $q(z) = b_0 + \underbrace{b_j}_{\substack{\text{insist that} \\ b_j \neq 0 \\ (\text{possibly } j=1)}} z^j + b_{j+1}z^{j+1} + \dots + b_{k-1}z^{k-1} + z^k$

A keypoint (again see §0.7): $z = x+iy \in \mathbb{C}$ $= re^{i\theta}$ has $z^m = r^m e^{im\theta}$

so as θ goes 0 to 2π,

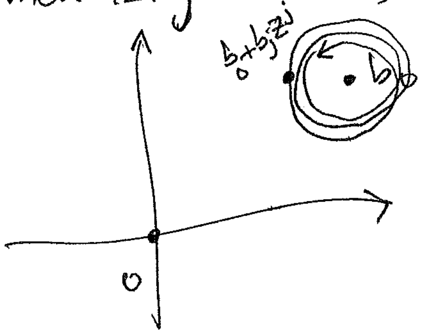


z travels once around, z^m travels m times around

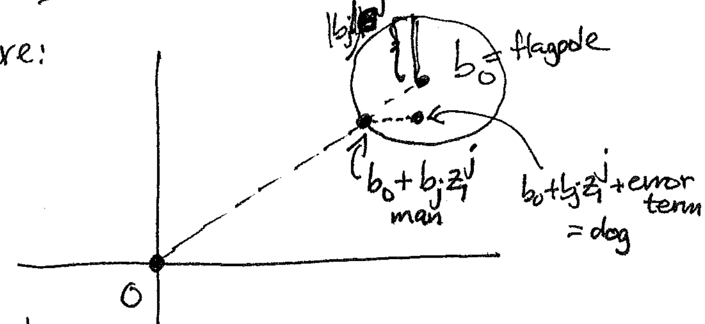
$$= \underbrace{b_0}_{\substack{\text{"flagpole"}}} + \underbrace{b_j z^j}_{\text{"man"}} + \underbrace{b_{j+1}z^{j+1} + \dots + b_{k-1}z^{k-1} + z^k}_{\text{"error term from" leash}} = \text{"dog's location"}$$

When $|z| = \rho$ is small, $b_0 + b_j z^j$ travels in a small circle around b_0, j times:

We'll try to find z_1 on this circle making $|q(z_1)| < |q(z_0)|$ with $|z_1| = \rho$



Pick ϵ very small, and pick z_1 as in this picture:



Then we'll have $|q(z_1)| < |b_0| = |q(z_0)|$

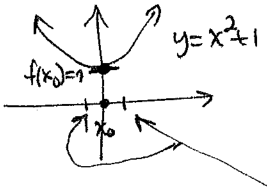
as long as $\underbrace{|b_{j+1}z^{j+1} + \dots + b_{k-1}z^{k-1} + z^k|}_{} \overset{?}{<} |b_j|\rho^j$

$$\leq |b_{j+1}|\rho^{j+1} + \dots |b_{k-1}|\rho^{k-1} + \rho^k$$

$$\leq \underbrace{\max\{|b_{j+1}|, \dots, |b_{k-1}|, 1\}}_{\text{call this } B :=} \cdot (k-j)\rho^{j+1}$$

So we need $B(k-j)\rho^{j+1} < |b_j|\rho^j$, i.e. $B(k-j)\rho < |b_j|$ or $\rho < \dfrac{|b_j|}{B(k-j)}$ ∎

Q: Where did this proof fail for $f(x) = x^2+1$ having no roots $x \in \mathbb{R}$?



It *does* achieve a minimum value $f(x_0) = 1$ at $x_0 = 0$.
But the "man" can't walk around the "flagpole" in a full circle, only at 2 points

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COR 1.6.14: A polynomial $p(z) = z^k + a_{k-1}z^{k-1} + \cdots + a_1 z^1 + a_0$ with $a_i \in \mathbb{C}$ and $k \geq 1$ has exactly k roots $r_1, \ldots, r_k$ in $\mathbb{C}$ (if you count with multiplicity), and factors as $p(z) = (z-r_1)(z-r_2)\cdots(z-r_k)$.

proof: Induct on k. The base case $k=1$ has $p(z) = z^1 + a_0 = z - r_1$ with $r_1 = a_0$.
In the inductive step, assume it for $k-1$, and given $p(z)$ of degree k, find some root r_1 using Fund'l Thm. Alg.
Use long division algorithm to write $p(z) = q(z)(z-r_1) + b$ for some $b \in \mathbb{C}$ and polynomial $q(z)$ of degree $k-1$, also monic, i.e. $q(z) = z^{k-1} + b_{k-2}z^{k-2} + \cdots$

$$\begin{array}{r|l} & q(z) = z^{k-1} + b_{k-2}z^{k-2} + \cdots \\ \hline z - r_1 & z^k + a_{k-1}z^{k-1} + \cdots + a_1 z + a_0 = p(z) \\ & \vdots \\ & \vdots \\ & \vdots \\ & b \end{array}$$

remainder b is of degree 0, i.e. $b \in \mathbb{C}$

However r_1 a root of $p(z)$ forces

$$0 = p(r_1) = q(r_1)\underbrace{(r_1 - r_1)}_{0} + b = b \quad \text{, i.e. } p(z) = q(z)(z-r_1)$$

Now apply induction to $q(z)$ ∎

10/10/2016 > What about irreducible factors of $p(x) = x^k + a_{k-1}x^{k-1} + \cdots + a_0$ with $a_i \in \mathbb{R}$ if we only allow real coefficients in the factors?

e.g. $x^4 - 1 = (x^2-1)(x^2+1)$

$= (x-1)(x+1)\underbrace{(x^2+1)}$ irreducible over $\mathbb{R}$

$\left(= (x-1)(x+1)(x-i)(x+i) \text{ over } \mathbb{C} \right)$