09/23/2016 Limits and Continuity

- Open sets, closed sets, etc...

We start with

distance in $\mathbb{R}^n$

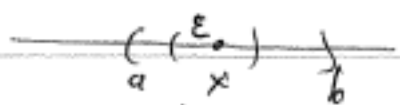
defn: open ball $B_\varepsilon(\underline{x}) = \{\underline{y} \in \mathbb{R}^n \mid |\underline{y} - \underline{x}| < \varepsilon\}$

Defn: A set $\mathcal{U} \subseteq \mathbb{R}^n$ is called <u>open</u>, if $\forall\, x \in \mathcal{U}$, $\exists\ \varepsilon > 0$ s.th. $B_\varepsilon(x) \subseteq \mathcal{U}$

"for every pt in $\mathcal{U}$, there is an open ball <u>in $\mathcal{U}$</u> centered at that pt"

Examples:

i) $(a,b) \subseteq \mathbb{R}$:



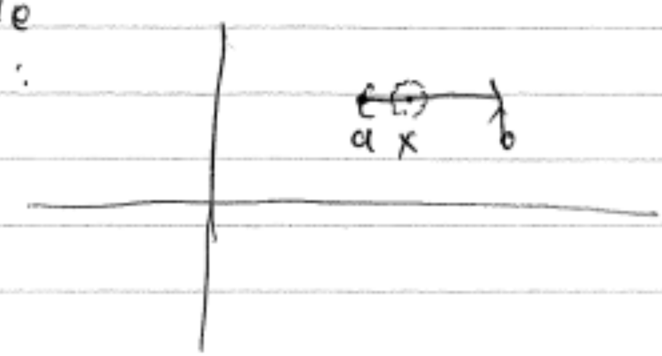
i.e. $\varepsilon = \min\{\frac{x-a}{2}, \frac{b-x}{2}\}$

ii) generic:

"boundary" pts <u>not</u> in $\mathcal{U}$

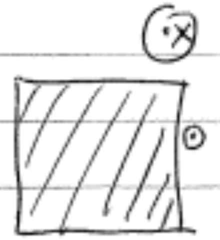
iii) non-example

$(a,b) \subseteq \mathbb{R}^2$:

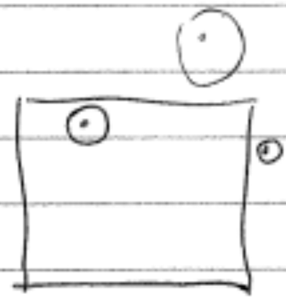


Defn: Closed sets: A set $C \subseteq \mathbb{R}^n$ is closed if its complement $\mathbb{R}^n - C$ is open.

Examples:

i) 

square (with inner part), indeed complement is open
=> each pt x has minimum distance from square.

ii) 

empty square. (same)

iii) non-example
$A = \{1, \frac{1}{2}, \dots, \frac{1}{n}, \dots\}$

Here $0 \in A^c$ (complement) but for any $\varepsilon > 0$
$B_\varepsilon(0) \cap A$ (i.e. $B_\varepsilon(0) \not\subset A^c$ so A^c is not open, so A is not closed)

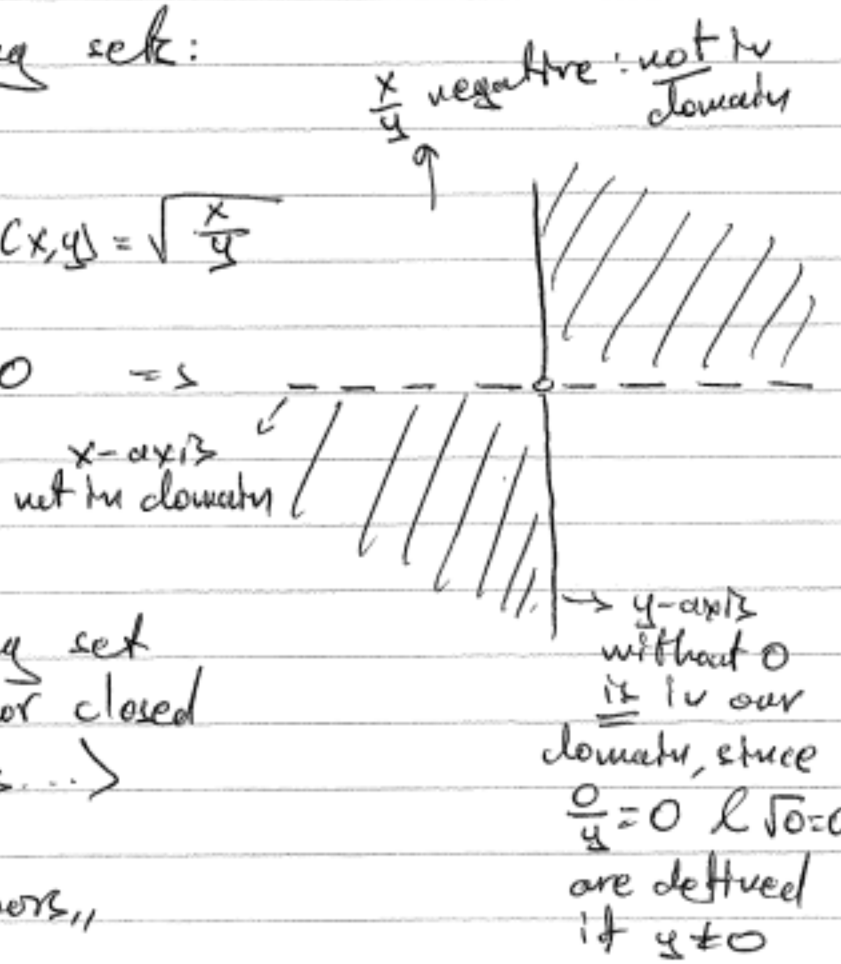
Enough to show that $\forall \varepsilon > 0$, $\exists\, n$ (number) s.th. $\frac{1}{n} < \varepsilon$. Pick $n > \frac{1}{\varepsilon}$ (Archimedean property)

Naturally appearing sets:

domains →

Example 1.5.5: $f(x,y) = \sqrt{\frac{x}{y}}$

We need $y \neq 0$, $\frac{x}{y} \geq 0$ ⇒



A naturally occuring set is neither open nor closed (look at two axes...)

⇒ "Sets are not doors"

⇒ Q: Example of a set that is both open and closed? → $\mathbb{R}$

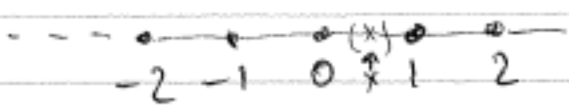
Why is $\emptyset$ open? → ~~empty states~~ "vacuous truth"

Rational numbers: ($\mathbb{Q} \subseteq \mathbb{R}$)

is $\mathbb{Q}$ open? No: any (a,b) contains irrationals

is $\mathbb{Q}$ closed? No: any (a,b) contains rationals

However $\mathbb{Z} \subseteq \mathbb{R}$ is closed.



(infinite sets of distinct pts can be closed..)

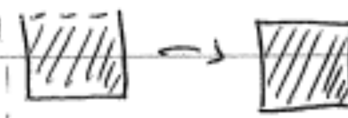
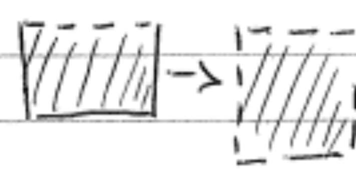
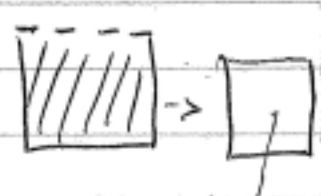
Q: An open set $\mathcal{U} \subset \mathbb{R}$ s.th. $\mathbb{Q} \subseteq \mathcal{U}$
and $\mathcal{U} \neq \mathbb{R}$

A: $\mathbb{R} - \{\pi\}$ → any irrational

Q: An open set $\mathcal{U} \subset \mathbb{R}$ s.th. $\mathbb{Q} \subseteq \mathcal{U}$
and $\mathcal{U}$ has "finite length"?

A: use balls with radius $\frac{1}{2^n}$ (b/c $\sum_{n=0}^{\infty} \frac{1}{2^n} = 2$)
Need: Rational numbers are countable ...

Three more concepts:

Term	Idea	Definition	Example
closure: of $\mathcal{U}$	The smallest closed set that contains $\mathcal{U}$	$\bar{\mathcal{U}} := \{ \underline{x} \in \mathbb{R}^n \mid B_r(\underline{x}) \cap \mathcal{U} \neq \emptyset \ \forall r > 0 \}$	
interior: of $\mathcal{U}$	The largest open set that is inside $\mathcal{U}$	$\mathcal{U}^\circ := \{ \underline{x} \in \mathbb{R}^n \mid \exists r > 0$ with $B_r(\underline{x}) \subseteq \mathcal{U} \}$	
boundary: of $\mathcal{U}$	Those pts that might be added/removed when taking closure/interior	$\partial \mathcal{U} := \{ \underline{x} \in \mathbb{R}^n \mid B_r(\underline{x}) \cap \mathcal{U} \neq \emptyset$ and $B_r(\underline{x}) \cap \mathcal{U}^c \neq \emptyset$ for <u>all</u> $r > 0 \}$ ("complement")	 empty inside

- Limits of sequences:

Defn: A sequence $\{\underline{a_n}\}_{n=1}^{\infty}$ (with $\underline{a_n} \in \mathbb{R}^k$) converges to a point $\underline{a} \in \mathbb{R}^k$ if for every $\varepsilon > 0$ there is some number M s.th. $\forall m > M$ $|\underline{a_m} - \underline{a}| < \varepsilon$

(For a challenge ε, I have to provide an answer M...)

Examples: i) $a_n = \frac{1}{n} \in \mathbb{R}$: $a_n \to 0$.

I need to show that for all $\varepsilon > 0$, $\exists M$ s.th for all $n > M$, I have $|\frac{1}{n} - 0| < \varepsilon$. Pick any $M > \frac{1}{\varepsilon}$.
Same as $\frac{1}{n} < \varepsilon \iff n > \frac{1}{\varepsilon}$.

=> Rethink our example: $\{\frac{1}{n}\}$ was not a closed set because it did not contain all of its limit points.

ii) $\{a_n\}_{n=1}^{\infty}$: $a_n = \begin{pmatrix} \frac{1}{n} \\ \frac{n}{n+1} \end{pmatrix} \in \mathbb{R}^2$.

$a_n \to \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ clearly! But how to prove?

$$\left|\underline{a_n} - \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right| < \varepsilon \iff \sqrt{\frac{1}{n^2} + \frac{(-1)^2}{(n+1)^2}} < \varepsilon$$

some not so difficult algebra should work:

$$\sqrt{\frac{1}{n^2} + \frac{1}{(n+1)^2}} < \sqrt{\frac{2}{n^2}} = \frac{\sqrt{2}}{n}$$ easy to force to be $< \varepsilon$...

Better way: I would like theorem to say:

$$\frac{1}{n} \to 0 \ \& \ \frac{n}{n+1} \to 1 \Rightarrow \begin{pmatrix} \frac{1}{n} \\ \frac{n}{n+1} \end{pmatrix} \to \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

This is true indeed!:

If $(\underline{a_n}) \in \mathbb{R}^k$ and $\underline{a_n} = ((\underline{a_n})_1, .., (\underline{a_n})_k)$
and $\underline{a} = (a_1, .., a_k)$,
I am given that $(\underline{a_n})_i \to a_i$
and I want to show that $(\underline{a_n}) \to \underline{a}$.

The assumption means that for any given
$\varepsilon_1, .., \varepsilon_k$, I can find $M_1, .., M_k$ so that

- $|(a_{m_1})_1 - a_1| < \varepsilon_1 \quad \forall m_1 > M_1$ — $(\underline{a_{m_1}})_1$: index of m_1, 1st coordinate of $\underline{a_{m_1}}$
- $|(a_{m_2})_2 - a_2| < \varepsilon_2 \quad \forall m_2 > M_2$

$\vdots$

- $|(a_{m_k})_k - a_k| < \varepsilon_k \quad \forall m_k > M_k$.

To show that $(\underline{a_n}) \to \underline{a}$, I want for any $\varepsilon > 0$
an M s.th. $\forall m > M$

$$|(\underline{a_m}) - \underline{a}| < \varepsilon \iff \sqrt{[(\underline{a_m})_1 - a_1]^2 + \dots + [(\underline{a_m})_k - a_k]^2} < \varepsilon.$$

Choose $M > \max\{M_1, .., M_k\}$ and you have

$$\sqrt{\varepsilon_1^2 + \dots + \varepsilon_k^2} < \varepsilon.$$

That is, for a given $\varepsilon > 0$, I have to find k-many small positive ε_i's such that

$$\sqrt{\varepsilon_1^2 + \cdots + \varepsilon_k^2} < \varepsilon.$$

This is definitely do-able!
Pick $\varepsilon_i < \frac{\varepsilon}{\sqrt{k}}$ for example.