(31)
9/25/2016 →

Section 1.5 is long, but important, and has a brief point:

multivariable limits are (almost) just like single variable limits!

Recall limits of sequences $(a_m)_{m=1}^{\infty} = (a_1, a_2, \ldots)$ in $\mathbb{R}^1$

e.g. $a_m = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^m}$ has $\lim_{m\to\infty} a_m = 2$

$a_1 = \frac{3}{2}$

$a_2 = \frac{7}{4}$

$a_3 = \frac{15}{8}$

$\vdots$

$a_m = \frac{2^{m+1}-1}{2^m} = 2 - \frac{1}{2^m}$

since $\forall \epsilon > 0\ \exists M$ such that

$m > M \Rightarrow \underbrace{|a_m - 2|}_{} < \epsilon$

$= |(2 - \frac{1}{2^m}) - 2|$

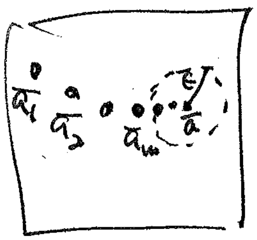
$= \frac{1}{2^m}$

(so $M = \log_2(\frac{1}{\epsilon})$ would work)

Similarly...

DEF'N 1.5.12: $(\bar{a}_m)_{m=1}^{\infty} = (\bar{a}_1, \bar{a}_2, \ldots)$ in $\mathbb{R}^n$ converges to $\bar{a} \in \mathbb{R}^n$

(written $\lim_{m\to\infty} \bar{a}_m = \bar{a}$)



if $\forall \epsilon > 0\ \exists M$ such that $m > M \Rightarrow |\bar{a}_m - \bar{a}| < \epsilon$ (or "$\bar{a}_m$ is in $B_\epsilon(\bar{a})$")

e.g. Theo showed $\bar{a}_m = \begin{pmatrix} 1/m \\ 1/(m+1) \end{pmatrix} \in \mathbb{R}^2$ has $\lim_{m\to\infty} \bar{a}_m = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \bar{a}$

- Convergence in $\mathbb{R}^n$ is coordinatewise:

PROP 1.5.13: $\lim_{m\to\infty} \bar{a}_m = \bar{a} \iff \lim_{m\to\infty} (a_m)_1 = a_1$, $\ldots$, $\lim_{m\to\infty} (a_m)_n = a_n$

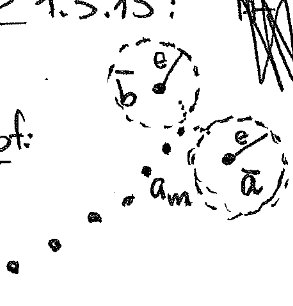
(not hard; see book)

where $\bar{a}_m = \begin{pmatrix} (a_m)_1 \\ \vdots \\ (a_m)_n \end{pmatrix}$, $\bar{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$

- Limits are unique (when they exist):

PROP 1.5.15: If $(\bar{a}_m)_{m=1}^{\infty}$ converges to $\bar{a}$, and also to $\bar{b}$, then $\bar{a} = \bar{b}$.

proof:



Suppose not, say $\bar{b} \neq \bar{a}$. Pick $\epsilon = \frac{|\bar{b} - \bar{a}|}{4}$

and $\exists$ some M_1 with $|\bar{a}_m - \bar{a}| < \epsilon$ for $m > M_1$

M_2 with $|\bar{a}_m - \bar{b}| < \epsilon$ for $m > M_2$,

but then for $m > \max(M_1, M_2)$ one has

$|\bar{b} - \bar{a}| \underset{\uparrow \text{triangle inequality}}{\leq} |\bar{b} - \bar{a}_m| + |\bar{a}_m - \bar{a}| < \epsilon + \epsilon = 2\epsilon = 2\frac{|\bar{b} - \bar{a}|}{4} < |\bar{b} - \bar{a}|$. Contradiction ∎

- There are various expected limit laws:

THM 1.5.16:

(again, not very hard, but let's skip it)

$$\lim_{m\to\infty} (\bar{a}_m + \bar{b}_m) = \lim_{m\to\infty} \bar{a}_m + \lim_{m\to\infty} \bar{b}_m \text{ if these exist}$$

$$\lim_{m\to\infty} c\,\bar{a}_m = c \lim_{m\to\infty} \bar{a} \quad \text{for } c\in\mathbb{R}, \text{ if this exists}$$

$$\lim_{m\to\infty} \bar{a}_m \cdot \bar{b}_m = \left(\lim_{m\to\infty} \bar{a}_m\right)\cdot\left(\lim_{m\to\infty} \bar{b}_m\right)$$

$$\lim_{m\to\infty} c_m\,\bar{a}_m = \bar{0} \quad \text{if } (\bar{a}_m)_{m=1}^{\infty} \text{ is bounded i.e. } \exists R\in\mathbb{R} \text{ with } |\bar{a}_m|\le R\ \forall m$$

and $\lim_{m\to\infty} c_m = 0$

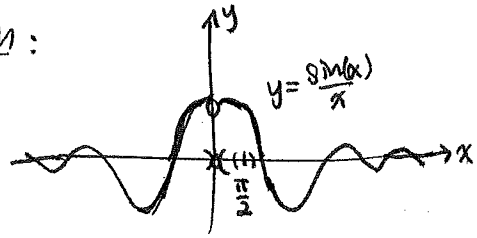
REMARK: Need the bounded hypothesis here to prevent things like $\lim_{m\to\infty} \underbrace{\frac{1}{m}}_{c_m} \underbrace{\begin{bmatrix} 2m \\ 3m^2 \end{bmatrix}}_{\bar{a}_m} = \begin{bmatrix} 2 \\ "\infty" \end{bmatrix} \neq \bar{0}$

(so $\lim_{m\to\infty} c_m = 0$)

Again, limits of functions are defined similarly to those for $f:\mathbb{R}^1 \to \mathbb{R}^1$

e.g. recall $f(x) = \frac{\sin(x)}{x}$ has natural domain $\mathcal{U} = \mathbb{R}^1 - \{0\}$ $\longleftrightarrow$ $\mathbb{R}^1$, 0

graph:



$\mathcal{U}$ is open: $\forall x\in\mathcal{U}\ \exists r>0$ with $B_r(x)\subset\mathcal{U}$

so $\mathcal{U} = \mathring{\mathcal{U}} :=$ the interior of $\mathcal{U}$

$\overline{\mathcal{U}} = \mathbb{R}^1$

$:=$ closure of $\mathcal{U}$

$= \{x\in\mathbb{R}^1 : \forall r>0,\ B_r(x)\cap\mathcal{U}\neq\emptyset\}$

$$\Rightarrow \partial\mathcal{U} = \overline{\mathcal{U}} \setminus \mathcal{U} = \{0\}$$

For $x_0\in\mathcal{U}$, e.g. $x_0 = \frac{\pi}{2}$, we can ask if $\lim_{x\to x_0} f(x)$ exists, e.g. $\lim_{x\to\frac{\pi}{2}} \frac{\sin(x)}{x} = \frac{\sin(\frac{\pi}{2})}{\frac{\pi}{2}} = \frac{1}{\pi/2} = \frac{2}{\pi}$

i.e. $\forall\epsilon>0\ \exists\delta>0$ with $|x-\frac{\pi}{2}|<\delta \Rightarrow \left|\frac{\sin(x)}{x} - \frac{2}{\pi}\right| < \epsilon$.

$= \lim_{x\to\frac{\pi}{2}^+} \frac{\sin(x)}{x} = \lim_{x\to\frac{\pi}{2}^-} \frac{\sin(x)}{x}$

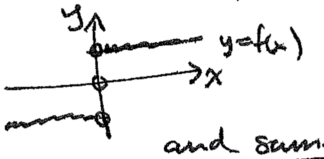
For $x_0\in\partial\mathcal{U}$, e.g. $x_0 = 0$, the question still makes sense:

does $\lim_{x\to 0} \frac{\sin(x)}{x}$ exist? (Yes, it is 1; not obvious but true in 1-variable calc)

$$\lim_{x\to 0^+} \frac{\sin(x)}{x} = \lim_{x\to 0^-} \frac{\sin(x)}{x}$$

e.g. $f(x) = \frac{|x|}{x} = \begin{Bmatrix} +1 \text{ if } x>0 \\ -1 \text{ if } x<0 \end{Bmatrix}$ has same natural domain $U = \mathbb{R}^1 - \{0\}$

$f: U \to \mathbb{R}^1$



and same questions make sense, but $\lim_{x \to 0} \frac{|x|}{x}$ does not exist,

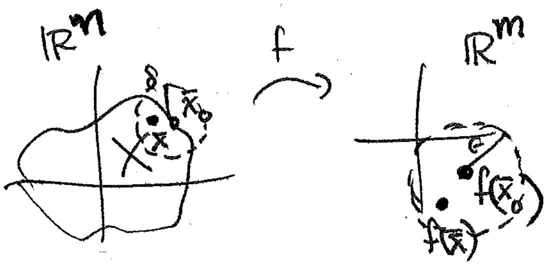
since $\lim_{x \to 0^+} \frac{|x|}{x} = +1$, $\lim_{x \to 0^-} \frac{|x|}{x} = -1$.

This issue is compounded in $\mathbb{R}^n$!

9/28/2016> The expected definitions and properties in $\mathbb{R}^n$...

DEF'N: For a subset $X \subset \mathbb{R}^n$ and function $\bar{f}: X \to \mathbb{R}^m$, and for any $\bar{x}_0 \in \overline{X}$ (= closure of X), say $\bar{f}$ has limit $\bar{a}$ at $\bar{x}_0$ (written $\lim_{\bar{x} \to \bar{x}_0} f(\bar{x}) = \bar{a}$)

if $\forall \epsilon > 0 \ \exists \delta > 0$ such that $\forall \bar{x} \in X$ $|\bar{x} - \bar{x}_0| < \delta \Rightarrow f(\bar{x}) - f(\bar{x}_0)| < \epsilon$.



PROP 1.5.21 (limits of functions are unique) If $\bar{a} = \lim_{\bar{x} \to \bar{x}_0} f(\bar{x})$, $\bar{b} = \lim_{\bar{x} \to \bar{x}_0} f(\bar{x})$ then $\bar{a} = \bar{b}$.

PROP 1.5.22 (limits of functions are componentwise) If $\bar{f}(\bar{x}) = \begin{pmatrix} f_1(\bar{x}) \\ \vdots \\ f_n(\bar{x}) \end{pmatrix} \in \mathbb{R}^m$, $\bar{x} \in \mathbb{R}^n$

then $\lim_{\bar{x} \to \bar{x}_0} \bar{f}(\bar{x}) = \underline{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix} \Leftrightarrow \begin{Bmatrix} \lim_{\bar{x} \to \bar{x}_0} f_1(\bar{x}) = a_1 \\ \vdots \\ \lim_{\bar{x} \to \bar{x}_0} f_m(\bar{x}) = a_m \end{Bmatrix}$