(3)

Certain subsets of $\mathbb{R}^n$ will be easiest to deal with, and arise frequently...

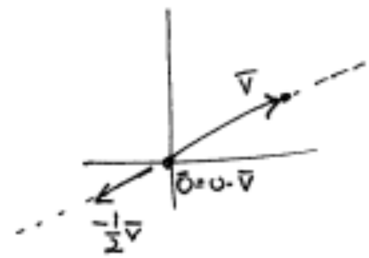
DEF'N: A (vector or linear) <u>subspace</u> $V \subset \mathbb{R}^n$ is a subset of vectors $\bar{v}$ which is

(a) nonempty $V \neq \emptyset$

(b) <u>closed</u> under $+$, that is, $\bar{v}_1, \bar{v}_2 \in V \underset{\text{implies}}{\Longrightarrow} \bar{v}_1 + \bar{v}_2 \in V$

(c) <u>closed</u> under scalar multiplication, that is,
$\bar{v} \in V,\ c \in \mathbb{R} \Rightarrow c\bar{v} \in V$

(In particular, (a),(c) $\Longrightarrow$ $\bar{0} = 0 \cdot \bar{v}\ \ \forall \bar{v} \in V$
$\in V$
so V contains the origin)



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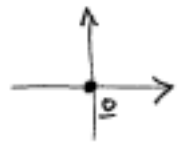
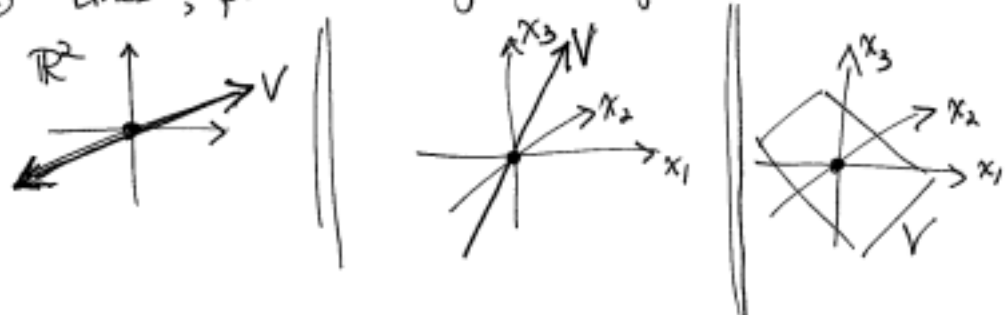
EXAMPLES:

① $\{\bar{0}\}$, $\mathbb{R}^n$ are trivial subspaces inside $\mathbb{R}^n$



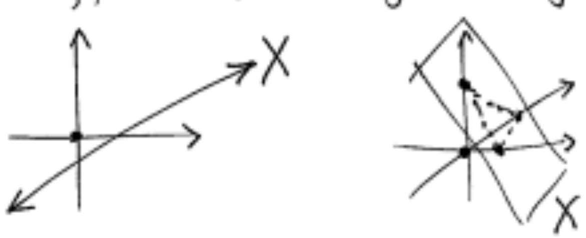
② Lines, planes through the origin



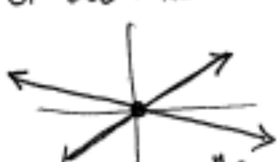
③ See EXERCISE 1.1.9 ⟵ got discussed a bit already in recitation; for fixed $w \in \mathbb{C}$,
$\{z \in \mathbb{C} : \mathrm{Re}(wz) = 0\}$

NON-EXAMPLES: $X \subset \mathbb{R}^n$

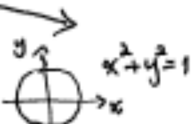
④ Lines, planes <u>not</u> through the origin (Why not?)



⑤ Union of two lines through origin (Why not?)



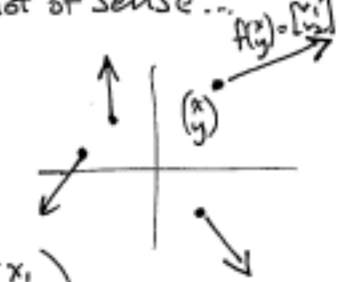
⑥ Unit circle



(Why not?) (<u>Lots</u> of non examples!)

(4)

REMARK: Later in the course, we encounter a situation where distinctions between points $\begin{pmatrix} x \\ y \end{pmatrix}$ versus vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ makes a lot of sense...



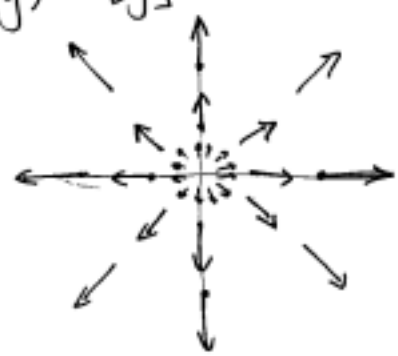
DEF'N: A vector field on $\mathbb{R}^n$ is a function

$$f : \underset{\substack{\text{domain} \\ \text{source}}}{\mathbb{R}^n} \longrightarrow \underset{\substack{\text{codomain} \\ \text{target}}}{\mathbb{R}^n}$$

sending a point $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \longmapsto \underset{\text{a vector}}{\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}} = f\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

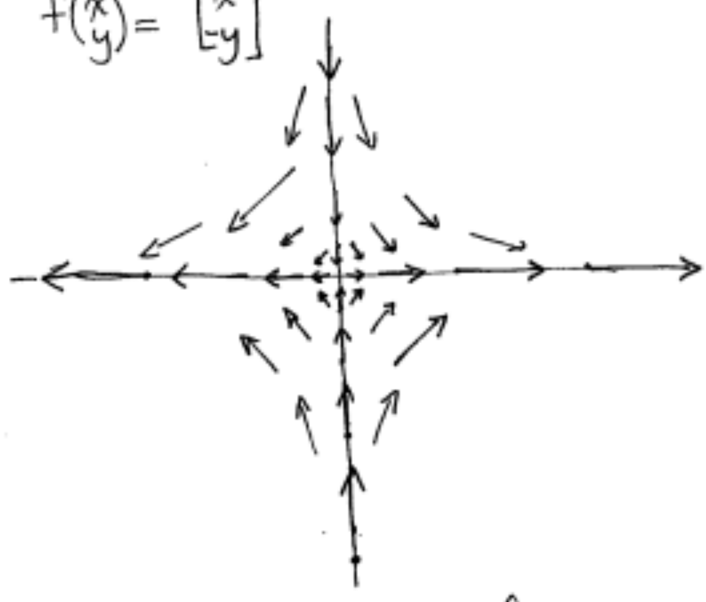
EXAMPLES in $\mathbb{R}^2$:

① $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$ the radial vector field



Tough to plot! Try googling for "vector field plotters"

② $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}$



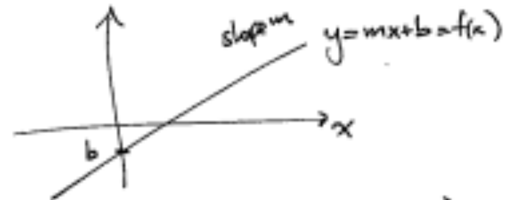
Later we calculate line integrals for curves through vector fields,
(2nd semester)
giving work done
or electrical flux

or one can solve differential equations to find flow lines through the vector field

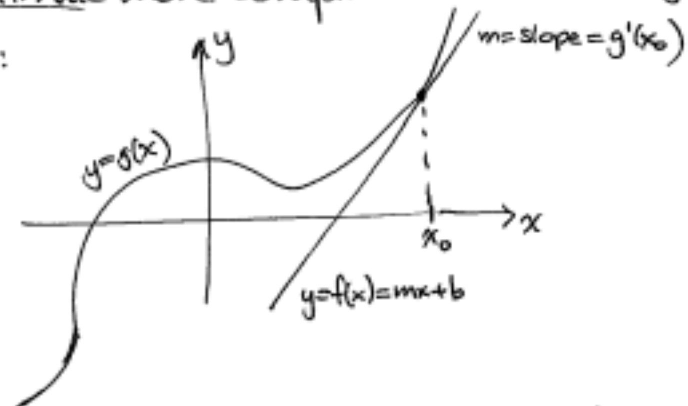
§1.2, 1.3 Matrices and linear transformations

IDEA: The easiest functions $f: \mathbb{R} \longrightarrow \mathbb{R}$, $x \longmapsto f(x)$ to graph $y=f(x)$ or solve equations $f(x)=c$

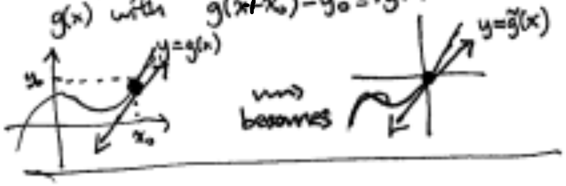
are ones of the form $f(x) = mx+b$



and we use them to approximate more complicated functions $g(x)$ by taking derivatives :



A different way to say this:
Translate the graph of $y=g(x)$ so that (x_0,y_0) on it goes through $\bar{0}$, by replacing $g(x)$ with $g(x+x_0)-y_0 =: \tilde{g}(x)$



The interesting part of $f(x)=mx+b$ is the $T(x)=mx$ (the linear part);

then $f(x) = T(x)+b$ is just adding on a fixed vector/point b.

Let's generalize $T(x)=mx$ to $\mathbb{R}^n$ ...

DEF'N: A function $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is called a linear transformation if it respects vector + and scalar multiplication,

i.e. $T(\bar{v}+\bar{w}) = T(\bar{v}) + T(\bar{w}) \quad \forall\, \bar{v},\bar{w} \in \mathbb{R}^n$

$T(c\bar{v}) = cT(\bar{v}) \quad \forall\, c \in \mathbb{R}$

$\Big(\Rightarrow T(c_1\bar{v}_1 + c_2\bar{v}_2 + \ldots + c_\ell \bar{v}_\ell) = c_1 T(\bar{v}_1) + \ldots + c_\ell T(\bar{v}_\ell)$

or $T\left(\sum_{i=1}^{\ell} c_i \bar{v}_i\right) = \sum_{i=1}^{\ell} c_i T(\bar{v}_i) \Big)$

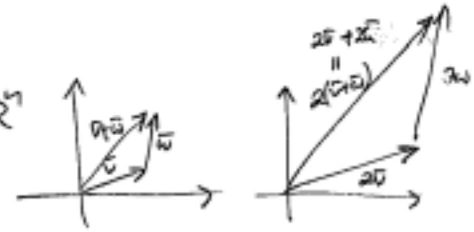
i.e. T respects linear combinations

(6)

EXAMPLES:

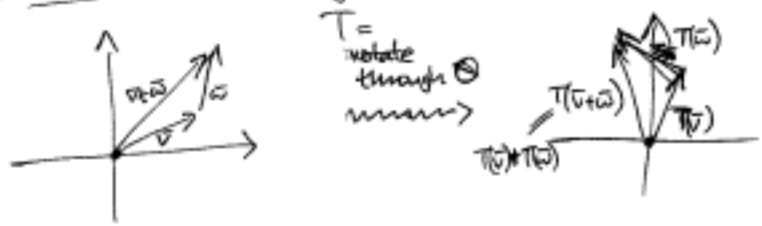
① Scalings $T(\bar{x}) = c\cdot\bar{x} = \begin{bmatrix} cx_1 \\ \vdots \\ cx_n \end{bmatrix}$, $T: \mathbb{R}^n \to \mathbb{R}^n$

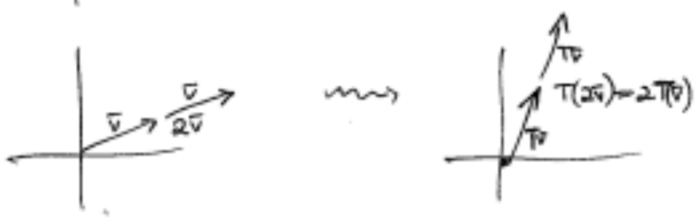
$$T(\bar{v}+\bar{w}) = c(\bar{v}+\bar{w}) = c\bar{v} + c\bar{w}$$



called rigid motions that fix $\bar{0}$

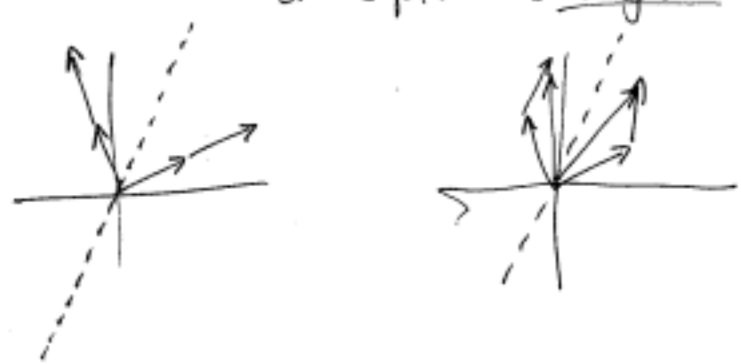
② Rotations through some angle θ about $\bar{0}$ in $\mathbb{R}^2$





③ Reflections in a line through $\bar{0}$ in $\mathbb{R}^2$

or a plane through $\bar{0}$ in $\mathbb{R}^3$



④ Projection orthogonally (perpendicularly) onto a line through $\bar{0}$ in $\mathbb{R}^2$ or a plane through $\bar{0}$ in $\mathbb{R}^3$

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NON-EXAMPLES:

⑤ Rotations not about $\bar{0}$, about some $\bar{x} \neq \bar{0}$, in $\mathbb{R}^2$

⑥ Reflections in lines not through $\bar{0}$ in $\mathbb{R}^2$

⑦ Projections orthogonally onto lines not through $\bar{0}$

⑧ $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$, since $(x+y)^2 \neq x^2 + y^2$

⑨ — " — $f(x) = \sin(x)$, since $\sin(x+y) \neq \sin(x) + \sin(y)$

⑩ $f(x) = mx + b$ with $b \neq 0$