

(117) To see this, note that  $h$  can't achieve its minimum on  $\partial U$  by the inequality (\*), so it achieves it at some  $\bar{x} \in U$  and then one must have  $0 = \frac{\partial h}{\partial x_j}(\bar{x}) = \sum_{i=1}^n \lambda_i (y_i - f_i(\bar{x})) \frac{\partial f_i}{\partial x_j}(\bar{x}) \quad \forall j=1, \dots, n$

$$\Rightarrow 0 = [J\bar{f}(\bar{x})](y - \bar{f}(\bar{x}))$$

def  $J\bar{f}(\bar{x}) \neq 0$   
 $\forall \bar{x} \in U$

$$\Rightarrow 0 = y - \bar{f}(\bar{x}), \text{ i.e. } \bar{f}(\bar{x}) = y.$$

12/13/2016  $\rightarrow$  Uniqueness of  $\bar{x}$  follows because we showed  $\forall \bar{x}, \bar{x}' \in U$  that  $|\bar{f}(\bar{x}) - \bar{f}(\bar{x}')| \geq \frac{1}{2} |\bar{x} - \bar{x}'|$ , so if  $\bar{f}(\bar{x}') = y = \bar{f}(\bar{x})$  then  $|\bar{x} - \bar{x}'| \leq \frac{1}{2} \cdot 0$  i.e.  $\bar{x}' = \bar{x}$ .  $\blacksquare$

So now if we define  ~~$V = \{\bar{x} \in U : \bar{f}(\bar{x}) \in W\}$~~   
 $V := \{\bar{x} \in U : \bar{f}(\bar{x}) \in W\}$

then as maps of sets, we have (from CLAIM 2)  $V \xrightleftharpoons[\bar{g}]{\bar{f}} W$  are 2-sided inverses

$$\begin{aligned} \bar{f} \circ \bar{g} &= 1_W \\ \bar{g} \circ \bar{f} &= 1_V \end{aligned}$$

(and important!)

Also, it is an easy exercise to check that since

$\bar{f}$  is continuous and  $W$  is open,  $V$  will also be open.

CLAIM 1 also shows  $\bar{g}$  is continuous, since  $\forall \epsilon > 0$ , if we choose  $\delta = \frac{\epsilon}{2}$

then we find that for  $\bar{y}_1, \bar{y}_2 \in W$  with  $|\bar{y}_1 - \bar{y}_2| < \delta = \frac{\epsilon}{2}$

the elements  $\bar{x}_1 = \bar{g}(\bar{y}_1)$  have  $\bar{y}_1 = \bar{f}(\bar{x}_1)$  so by CLAIM 1,

$$\bar{x}_2 = \bar{g}(\bar{y}_2) \quad \bar{y}_2 = \bar{f}(\bar{x}_2)$$

$$\begin{aligned} |f(\bar{x}_1) - f(\bar{x}_2)| &\geq \frac{1}{2} |\bar{x}_1 - \bar{x}_2| \\ &= \frac{1}{2} |\bar{g}(\bar{y}_1) - \bar{g}(\bar{y}_2)| \\ \frac{\epsilon}{2} &> |\bar{y}_1 - \bar{y}_2| \\ &\Rightarrow |\bar{g}(\bar{y}_1) - \bar{g}(\bar{y}_2)| < \epsilon. \end{aligned}$$

It remains to show  $\bar{g}$  is differentiable at every  $\bar{y} \in W$ .

In fact, we'll check that if  $\bar{g}(\bar{y}) = \bar{x} \in V$

(so  $\bar{f}(\bar{x}) = \bar{y}$ ), and if  $A := D\bar{f}(\bar{x})$

then  $\bar{A}' = D\bar{g}(\bar{y})$ , as we'd expect from chain rule.

(118) Thus we need to show

$$\lim_{k \rightarrow \bar{0}} \frac{|\bar{g}(\bar{y}+k) - \bar{g}(\bar{y}) - \bar{A}'k|}{|k|} \stackrel{?}{=} 0$$

let  $\bar{g}(\bar{y}+k) =: \bar{x}+\bar{h}$ , so  $\bar{f}(\bar{x}+\bar{h}) = \bar{y}+k$   
 $f(\bar{x}) = \bar{y}$   
 i.e.  $k = \bar{f}(\bar{x}+\bar{h}) - f(\bar{x})$

AND

$\bar{h} = \bar{g}(\bar{y}+k) - \bar{g}(\bar{y})$   
 so  $\bar{h} \rightarrow \bar{0}$   
 as  $k \rightarrow \bar{0}$   
 (by continuity of  $\bar{g}$ ,  
 already checked)

$$\lim_{k \rightarrow \bar{0}} \frac{|\bar{x}+\bar{h} - \bar{x} - \bar{A}'k|}{|k|}$$

$$= \lim_{k \rightarrow \bar{0}} \frac{|\bar{h} - \bar{A}'k|}{|k|} = \lim_{k \rightarrow \bar{0}} \frac{|\bar{A}'(\bar{A}k - k)|}{|k|}$$

$$\text{Now } \frac{|\bar{A}'(\bar{A}k - k)|}{|k|} \leq \|\bar{A}'\| \frac{|\bar{k} - \bar{A}\bar{h}|}{|k|} = \|\bar{A}'\| \cdot \frac{|\bar{k} - \bar{A}\bar{h}|}{|\bar{h}|} \cdot \frac{|\bar{h}|}{|k|}$$

$$= \underbrace{\|\bar{A}'\|}_{\text{constant}} \cdot \left( \frac{|\bar{f}(\bar{x}+\bar{h}) - \bar{f}(\bar{x}) - \bar{A}\bar{h}|}{|\bar{h}|} \right) \cdot \left( \frac{|\bar{x}+\bar{h} - \bar{x}|}{|\bar{f}(\bar{x}+\bar{h}) - \bar{f}(\bar{x})|} \right)$$

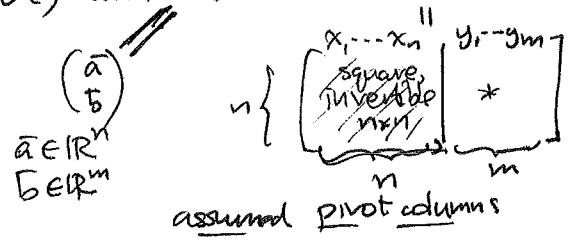
approaches 0  
 as  $k \rightarrow \bar{0}$   
 (since then  $\bar{h} \rightarrow \bar{0}$ , also  
 by def'n of  $D\bar{f}(\bar{x}) = \bar{A}$ )

$\leq 2$  by CLAIM 1

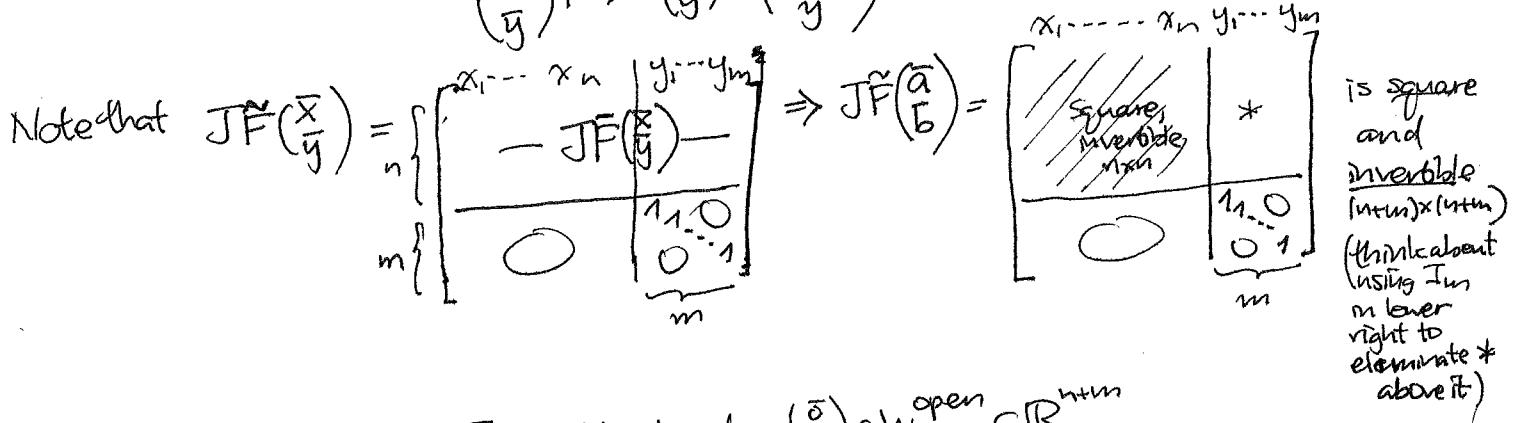
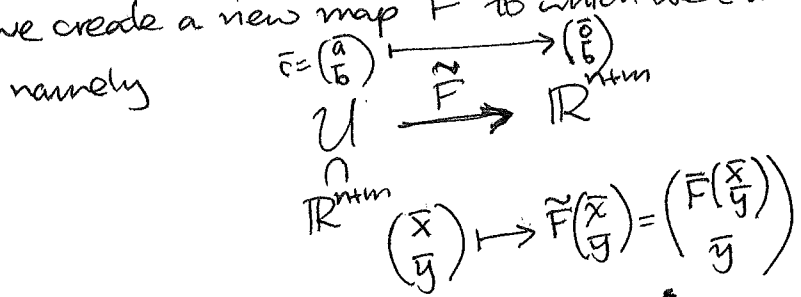
approaches 0 as  $k \rightarrow \bar{0}$  ▣

proof of Implicit Function Thm:

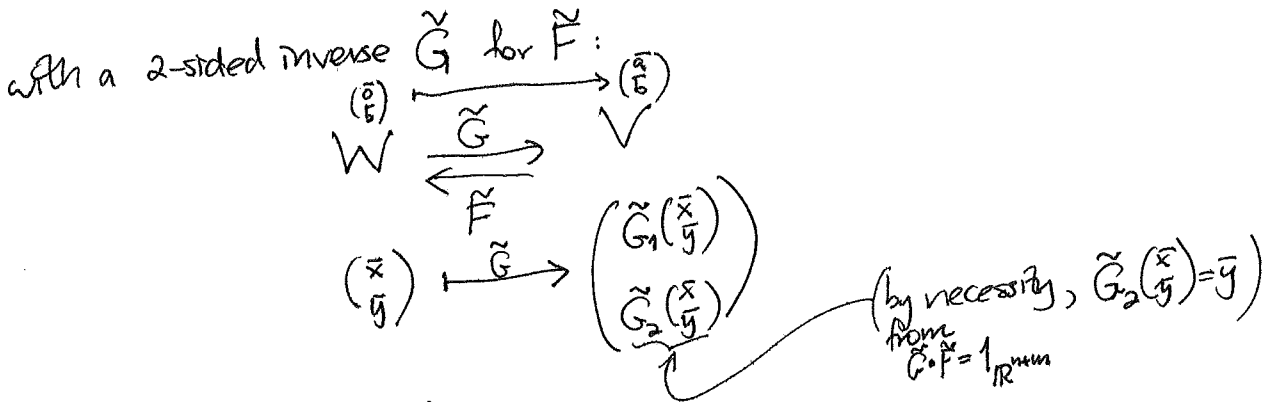
Given  $U \subset \mathbb{R}^{n+m}$  open  $F: U \rightarrow \mathbb{R}^m$  in  $C^1(U)$  and  $\bar{c} \in U$  with  $DF(\bar{c}): \mathbb{R}^{n+m} \rightarrow \mathbb{R}^m$  of full rank



we create a new map  $\tilde{F}$  to which we can apply Inverse Function Thm,



Thus by Inverse Fn. Thm,  $\exists$  neighborhoods  $(\bar{0}, \bar{b}) \in W \subset \mathbb{R}^{n+m}$  open  $\subset \mathbb{R}^{n+m}$   
 $\bar{c} = (\bar{a}, \bar{b}) \in V \subset \mathbb{R}^{n+m}$  open  $\subset \mathbb{R}^{n+m}$



Define  $\bar{g}(\bar{y}) = \tilde{G}_1(\bar{0}, \bar{y})$ .

Then we claim  $F(\bar{g}(\bar{y})) = \bar{0}$ , as desired, since

$$\begin{pmatrix} \bar{0} \\ \bar{y} \end{pmatrix} = \tilde{F} \circ \tilde{G} \begin{pmatrix} \bar{0} \\ \bar{y} \end{pmatrix} = \tilde{F} \begin{pmatrix} \bar{g}(\bar{y}) \\ \bar{y} \end{pmatrix} = \begin{pmatrix} F(\bar{g}(\bar{y}), \bar{y}) \\ \bar{y} \end{pmatrix}$$

and this holds  $\forall \begin{pmatrix} \bar{0} \\ \bar{y} \end{pmatrix}$  in  $W \subset \mathbb{R}^{n+m}$ , so it holds for all  $\bar{y}$  in an open neighborhood of  $\bar{b}$  in  $\mathbb{R}^m$ . Also  $\bar{g}$  is diffeable since  $\tilde{G}_1$  was (since  $\tilde{G}$  was)