To see this, note that h can't achieve its minimum on $\partial\mathcal{U}$ by the inequality $(*)$, so it achieves it at some $\bar{x} \in \mathcal{U}$ and then one must have $0 = \frac{\partial h}{\partial x_j}(\bar{x}) = \sum_{i=1}^{n} 2(y_i - f_i(\bar{x}))\frac{\partial f_i}{\partial x_j}(\bar{x}) \quad \forall j=1,\dots,n$

$$\Longrightarrow \quad \bar{0} = \left[Jf(\bar{x})\right](\bar{y} - \bar{f}(\bar{x}))$$

$\det Jf(\bar{x}) \neq 0$
$\forall \bar{x} \in \mathcal{U}$

$$\Longrightarrow \quad \bar{0} = \bar{y} - \bar{f}(\bar{x}), \quad \text{i.e. } \bar{f}(\bar{x}) = \bar{y}.$$

12/13/2016 Uniqueness of $\bar{x}$ follows because we showed $\forall \bar{x}, \bar{x}' \in \mathcal{U}$ that $|\bar{f}(\bar{x}) - \bar{f}(\bar{x}')| \geq \frac{1}{2}|\bar{x} - \bar{x}'|$, so if $\bar{f}(\bar{x}') = \bar{y} = \bar{f}(\bar{x})$ then $|\bar{x} - \bar{x}'| \leq \frac{1}{2}\cdot 0$ i.e. $\bar{x}' = \bar{x}$. ∎

So now if we define $V := \{\bar{x} \in \mathcal{U} : \bar{f}(\bar{x}) \in W\}$

then as maps of sets, we have (from CLAIM 2) $V \underset{\bar{g}}{\overset{\bar{f}}{\rightleftarrows}} W$ are 2-sided inverses
$\bar{f} \circ \bar{g} = 1_W$
$\bar{g} \circ \bar{f} = 1_V$

Also, it is an easy (and important!) exercise to check that since $\bar{f}$ is continuous and W is open, V will also be open.

CLAIM 1 also shows $\underline{\bar{g} \text{ is continuous}}$, since $\forall \epsilon > 0$, if we choose $\delta = \frac{\epsilon}{2}$ then we find that for $\bar{y}_1, \bar{y}_2 \in W$ with $|\bar{y}_1 - \bar{y}_2| < \delta = \frac{\epsilon}{2}$ the elements $\bar{x}_1 = \bar{g}(\bar{y}_1)$, $\bar{x}_2 = \bar{g}(\bar{y}_2)$ have $\bar{y}_1 = \bar{f}(\bar{x}_1)$, $\bar{y}_2 = \bar{f}(\bar{x}_2)$ so by CLAIM 1,

$$\frac{\epsilon}{2} > |\bar{y}_1 - \bar{y}_2| = |\bar{f}(\bar{x}_1) - \bar{f}(\bar{x}_2)| \geq \frac{1}{2}|\bar{x}_1 - \bar{x}_2| = \frac{1}{2}|\bar{g}(\bar{y}_1) - \bar{g}(\bar{y}_2)|$$

$$\Rightarrow |\bar{g}(\bar{y}_1) - \bar{g}(\bar{y}_2)| < \epsilon.$$

It remains to show $\bar{g}$ is $\underline{\text{differentiable}}$ at every $\bar{y} \in W$.

In fact, we'll check that if $\bar{g}(\bar{y}) = \bar{x} \in V$ (so $\bar{f}(\bar{x}) = \bar{y}$), and if $A := D\bar{f}(\bar{x})$ then $A^{-1} = D\bar{g}(\bar{y})$, as we'd expect from chain rule.

(118) Thus we need to show

$$\lim_{\bar{k}\to\bar{0}} \frac{|\bar{g}(\bar{y}+\bar{k})-\bar{g}(\bar{y})-\bar{A}^{1}\bar{k}|}{|\bar{k}|} \overset{?}{=} 0$$

let $\bar{g}(\bar{y}+\bar{k}) =: \bar{x}+\bar{h}$, so $\bar{f}(\bar{x}+\bar{h}) = \bar{y}+\bar{k}$
$f(\bar{x}) = \bar{y}$
i.e. $\bar{k} = \bar{f}(\bar{x}+\bar{h}) - f(\bar{x})$

AND

$\bar{h} = \bar{g}(\bar{y}+\bar{k}) - \bar{g}(\bar{y})$
so $\bar{h} \to \bar{0}$
as $\bar{k} \to \bar{0}$
(by continuity of $\bar{g}$, already checked)

$$\lim_{\bar{k}\to\bar{0}} \frac{|(\bar{x}+\bar{h})-\bar{x}-\bar{A}^{1}\bar{k}|}{|\bar{k}|}$$

$$= \lim_{\bar{k}\to\bar{0}} \frac{|\bar{h}-A^{-1}\bar{k}|}{|\bar{k}|} = \lim_{\bar{k}\to\bar{0}} \frac{|\bar{A}^{1}(A\bar{h}-\bar{k})|}{|\bar{k}|}$$

Now $$\frac{|\bar{A}^{1}(A\bar{h}-\bar{k})|}{|\bar{k}|} \le |A^{-1}| \frac{|\bar{k}-A\bar{h}|}{|\bar{k}|} = |A^{-1}| \cdot \frac{|\bar{k}-A\bar{h}|}{|\bar{h}|} \cdot \frac{|\bar{h}|}{|\bar{k}|}$$

$$\underbrace{= \underbrace{|A^{-1}|}_{\text{constant}} \cdot \underbrace{\frac{|\bar{f}(\bar{x}+\bar{h})-\bar{f}(\bar{x})-A\bar{h}|}{|\bar{h}|}}_{\substack{\text{approaches } 0 \\ \text{as } \bar{k}\to\bar{0} \\ (\text{since then } \bar{h}\to\bar{0}\text{, also}) \\ \text{by def'n of } D\bar{f}(\bar{x})=A}} \cdot \underbrace{\frac{|(\bar{x}+\bar{h})-\bar{x}|}{|f(\bar{x}+\bar{h})-f(\bar{x})|}}_{\le 2 \text{ by CLAIM 1}}}_{\text{approaches } 0 \text{ as } \bar{k}\to\bar{0}}$$

∎

proof of Implicit Function Thm:

Given $\underset{\mathbb{R}^{n+m}}{\underset{\cap}{\mathcal{U}^{open}}} \xrightarrow{\bar F} \mathbb{R}^n$ in $C^1(\mathcal{U})$ and $\bar c \in \mathcal{U}$ with $D\bar F(\bar c): \mathbb{R}^{n+m} \to \mathbb{R}^n$ of full rank

$$\bar c = \begin{pmatrix} \bar a \\ \bar b \end{pmatrix}, \quad \bar a \in \mathbb{R}^n, \ \bar b \in \mathbb{R}^m$$

$$D\bar F(\bar c) = \begin{array}{cc} & \begin{array}{cc} x_1 \cdots x_n & y_1 \cdots y_m \end{array} \\ n\{ & \left[\begin{array}{c|c} \text{square, invertible } n\times n & * \end{array}\right] \\ & \begin{array}{cc} \underbrace{\qquad\qquad}_{n} & \underbrace{\qquad}_{m} \end{array} \end{array}$$

assumed pivot columns

we create a new map $\tilde F$ to which we can apply Inverse Function Thm, namely

$$\bar c = \begin{pmatrix} \bar a \\ \bar b \end{pmatrix} \longmapsto \begin{pmatrix} \bar 0 \\ \bar b \end{pmatrix}$$

$$\underset{\mathbb{R}^{n+m}}{\underset{\cap}{\mathcal{U}}} \xrightarrow{\tilde F} \mathbb{R}^{n+m}$$

$$\begin{pmatrix} \bar x \\ \bar y \end{pmatrix} \longmapsto \tilde F \begin{pmatrix} \bar x \\ \bar y \end{pmatrix} = \begin{pmatrix} \bar F\begin{pmatrix} \bar x \\ \bar y \end{pmatrix} \\ \bar y \end{pmatrix}$$

Note that
$$J\tilde F\begin{pmatrix} \bar x \\ \bar y \end{pmatrix} = \begin{array}{cc} & \begin{array}{cc} x_1 \cdots x_n & y_1 \cdots y_m \end{array} \\ \begin{array}{c} n\{ \\ m\{ \end{array} & \left[\begin{array}{c|c} \multicolumn{2}{c}{— \ J\bar F\begin{pmatrix} \bar x \\ \bar y \end{pmatrix} \ —} \\ \hline 0 & \begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} \end{array}\right] \\ & \begin{array}{cc} & \underbrace{\qquad}_{m} \end{array} \end{array} \Rightarrow J\tilde F\begin{pmatrix} \bar a \\ \bar b \end{pmatrix} = \begin{array}{c} \begin{array}{cc} x_1 \cdots x_n & y_1 \cdots y_m \end{array} \\ \left[\begin{array}{c|c} \text{square, invertible } n\times n & * \\ \hline 0 & \begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} \end{array}\right] \\ \begin{array}{cc} & \underbrace{\qquad}_{m} \end{array} \end{array}$$

is square and invertible $(n+m)\times(n+m)$ (think about using I_m in lower right to eliminate $*$ above it)

Thus by Inverse Fn. Thm, $\exists$ neighborhoods $\begin{pmatrix} \bar 0 \\ \bar b \end{pmatrix} \in W^{open} \subset \mathbb{R}^{n+m}$, $\bar c = \begin{pmatrix} \bar a \\ \bar b \end{pmatrix} \in V^{open} \subset \mathbb{R}^{n+m}$

with a 2-sided inverse $\tilde G$ for $\tilde F$:

$$\begin{pmatrix} \bar 0 \\ \bar b \end{pmatrix} \longmapsto \begin{pmatrix} \bar a \\ \bar b \end{pmatrix}$$

$$W \underset{\tilde F}{\overset{\tilde G}{\rightleftarrows}} V$$

$$\begin{pmatrix} \bar x \\ \bar y \end{pmatrix} \overset{\tilde G}{\longmapsto} \begin{pmatrix} \tilde G_1\begin{pmatrix} \bar x \\ \bar y \end{pmatrix} \\ \tilde G_2\begin{pmatrix} \bar x \\ \bar y \end{pmatrix} \end{pmatrix}$$

(by necessity, $\tilde G_2\begin{pmatrix} \bar x \\ \bar y \end{pmatrix} = \bar y$ from $\tilde G \circ \tilde F = 1_{\mathbb{R}^{n+m}}$)

Define $\bar g(\bar y) := \tilde G_1\begin{pmatrix} \bar 0 \\ \bar y \end{pmatrix}$.

Then we claim $F\begin{pmatrix} \bar g(\bar y) \\ \bar y \end{pmatrix} = \bar 0$, as desired, since

$$\begin{pmatrix} \bar 0 \\ \bar y \end{pmatrix} = \tilde F \circ \tilde G \begin{pmatrix} \bar 0 \\ \bar y \end{pmatrix} = \tilde F \begin{pmatrix} \bar g(\bar y) \\ \bar y \end{pmatrix} = \begin{pmatrix} F\begin{pmatrix} \bar g(\bar y) \\ \bar y \end{pmatrix} \\ \bar y \end{pmatrix}$$

and this holds $\forall \begin{pmatrix} \bar 0 \\ \bar y \end{pmatrix}$ in $W^{open} \ni \begin{pmatrix} \bar 0 \\ \bar b \end{pmatrix}$, so it holds for all $\bar y$ in an open neighborhood of $\bar b$ in $\mathbb{R}^m$.
Also $\bar g$ is diff'ble since $\tilde G_1$ was (since $\tilde G$ was) ∎