Math 3593H Honors Math II
Quiz 1, Thursday Feb. 2, 2017

Instructions:
20 minutes, closed book and notes, no electronic devices.
There is one problem with three parts, worth a total of 20 points.

1. (9 points) Consider the set

$$M = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : x^5 + y^{50} = 3 - z^{500} \right\}.$$

(i) Prove that M is a manifold.

$$M = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : F\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \right\}$$

where $F\begin{pmatrix} x \\ y \\ z \end{pmatrix} = x^5 + y^{50} + z^{500} - 3$

$$\mathbb{R}^3 \xrightarrow{F} \mathbb{R}^1$$

$\Big\downarrow$ take derivative

$$\mathbb{R}^3 \xrightarrow[\#]{DF} \mathbb{R}^1$$

$$[5x^4 \quad 50y^{49} \quad 500z^{499}]$$

$DF(\bar{c})$ has full rank, i.e. rank 1,
as long as $\bar{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$ has either $5c_1^4 \neq 0$ i.e. $c_1 \neq 0$
or
$50c_2^{49} \neq 0$ i.e. $c_2 \neq 0$
or
$500c_3^{499} \neq 0$ i.e. $c_3 \neq 0$

i.e. as long as $\bar{c} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. But $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \notin M$.

So M is a manifold.

(ii) (2 points) What is the dimension of M as a manifold?

Since $M = \{F(\bar{z})=0\}$ for $\mathbb{R}^3 \xrightarrow{F} \mathbb{R}^1$
with $DF(\bar{c})$ of full rank 1 at all $\bar{c} \in M$,
its dimension is $3-1=2$.

(iii) (9 points) Write down a basis for the tangent space $T_{\left(\begin{smallmatrix}1\\1\\1\end{smallmatrix}\right)}M$.

$$T_{\left(\begin{smallmatrix}1\\1\\1\end{smallmatrix}\right)}M = \ker\left(\mathbb{R}^3 \xrightarrow[{[5\cdot 1^4 \;\; 50\cdot 1^{49} \;\; 500\cdot 1^{499}]}]{DF\left(\begin{smallmatrix}1\\1\\1\end{smallmatrix}\right)} \mathbb{R}^1\right)$$

$$= \ker\left(\mathbb{R}^3 \xrightarrow[{[5 \;\; 50 \;\; 500]}]{} \mathbb{R}^1\right)$$

row-reduction

$$\overset{\downarrow}{=} \ker\left(\begin{array}{ccc} x & y & z \\ \hline 1 & 10 & 100 \end{array}\right)$$

$$= \left\{ \begin{bmatrix} x\\y\\z \end{bmatrix} \in \mathbb{R}^3 \text{ of form } \begin{bmatrix} -10y-100z \\ y \\ z \end{bmatrix} \right\}$$

$$= \left\{ \begin{bmatrix} -10\\1\\0 \end{bmatrix} y + \begin{bmatrix} -100\\0\\1 \end{bmatrix} z : \underset{y,z\in\mathbb{R}}{} \right\}$$

, so it has basis $\left\{ \begin{bmatrix} -10\\1\\0 \end{bmatrix}, \begin{bmatrix} -100\\0\\1 \end{bmatrix} \right\}$