(119)

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EXAMPLES: ① $f\begin{pmatrix}x\\y\\z\end{pmatrix} = x^2+y^2+z^2 \rightsquigarrow \bar{F} = \operatorname{grad} f = \text{"}\begin{bmatrix}\partial/\partial x\\ \partial/\partial y\\ \partial/\partial z\end{bmatrix} f\text{"} = \begin{bmatrix}2x\\2y\\2z\end{bmatrix} = 2\begin{bmatrix}x\\y\\z\end{bmatrix}$

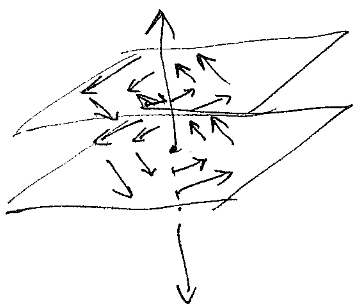
i.e. $\bar{F}$ conservative

has $\operatorname{curl}\bar{F} = \operatorname{curl}\operatorname{grad} f = \text{"}\begin{bmatrix}\partial/\partial x\\ \partial/\partial y\\ \partial/\partial z\end{bmatrix}\times\bar{F}\text{"} = 2\begin{bmatrix}\partial/\partial x\\ \partial/\partial y\\ \partial/\partial z\end{bmatrix}\times\begin{bmatrix}x\\y\\z\end{bmatrix}$

$= 2\begin{bmatrix}0\\0\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

i.e. $\bar{F}$ is irrotational

② $\bar{F}\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}-y\\x\\0\end{pmatrix}$ has $\operatorname{curl}\bar{F} = \text{"}\begin{bmatrix}\partial/\partial x\\ \partial/\partial y\\ \partial/\partial z\end{bmatrix}\times\begin{bmatrix}-y\\x\\0\end{bmatrix}\text{"} = \begin{bmatrix}0\\0\\2\end{bmatrix}$ pointing toward $\bar{e}_3$



$\operatorname{div}\bar{F}$ is easier to understand for $\bar{F} = \begin{pmatrix}F_1\\F_2\\F_3\end{pmatrix} : \mathbb{R}^3 \to \mathbb{R}^3$,

as it is measuring locally at $\bar{x}\in\mathbb{R}^3$ how much $\bar{F}$ flows "outward" (diverges) from $\bar{x}$:

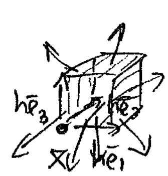
Letting $\Phi_{\bar{F}} = F_1\,dy\wedge dz + F_2\,dx\wedge dz + F_3\,dx\wedge dy$, the flux 2-form for $\bar{F}$

$\downarrow d$

then $d\Phi_{\bar{F}} = \operatorname{div}\bar{F}(\bar{x})\,dx\wedge dy\wedge dz$,

so $\operatorname{div}\bar{F}(\bar{x}) = \operatorname{div}\bar{F}(\bar{x})\,dx\wedge dy\wedge dz\left(P_{\bar{x}}(\bar{e}_1,\bar{e}_2,\bar{e}_3)\right) = d\Phi_{\bar{F}}\left(P_{\bar{x}}(\bar{e}_1,\bar{e}_2,\bar{e}_3)\right)$

$$= \lim_{h\to 0}\frac{1}{h^2}\underbrace{\int_{\partial P_{\bar{x}}(h\bar{e}_1,h\bar{e}_2,h\bar{e}_3)}\Phi_{\bar{F}}}_{\text{flux through } \partial P_{\bar{x}}(h\bar{e}_1,h\bar{e}_2,h\bar{e}_3) \text{ of } \bar{F}}$$



e.g. when $\bar{F}$ is electric field, $\operatorname{div}\bar{F}(\bar{x}) =$ charge density at $\bar{x}$ (=0 when no charges present)

$\bar{B}$ is magnetic field, $\operatorname{div}\bar{B}(\bar{x}) = 0$ since there are no magnetic charges

$\bar{F}$ is fluid velocity, $\operatorname{div}\bar{F}(\bar{x}) = 0$ when there are no sources around, and the fluid is incompressible (no local change in density possible).

§6.10 Green's, Stokes's, (Gauss's) Divergence Theorems

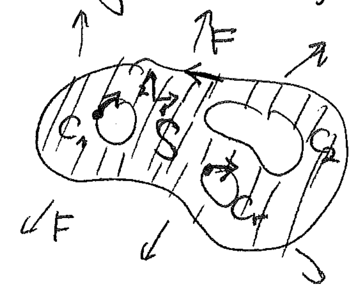
These are just the special cases of the general Stokes's Thm $\int_{\partial X}\varphi = \int_X d\varphi$

for flux integrals vs. line integrals in $\mathbb{R}^2, \mathbb{R}^3$

and volume vs. flux in $\mathbb{R}^3$

THM 6.10.2 (Green's Thm)

For a piece-with-boundary $S \subset \mathbb{R}^2$ having boundary curves $C_1, \dots, C_r$, and $\vec{F} = \begin{pmatrix} f(\vec{x}) \\ g(\vec{x}) \end{pmatrix} : \mathbb{R}^2 \to \mathbb{R}^2$ a vector field,



$$\int_S dW_{\vec{F}} = \int_{\partial S} W_{\vec{F}} = \sum_{i=1}^{r} \underbrace{\int_{C_i} W_{\vec{F}}}_{\text{using boundary orientations}}$$

$$\parallel \qquad\qquad\qquad \parallel$$

$$\int_S \underbrace{\left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)}_{\substack{\text{like the} \\ \text{2-dimensional} \\ \text{version of curl}(\vec{F}), \\ \text{having only an } \vec{e}_3\text{-component}}} dx \wedge dy \qquad\qquad \sum_{i=1}^{r} \int_{C_i} (f\,dx + g\,dy)$$

More generally,

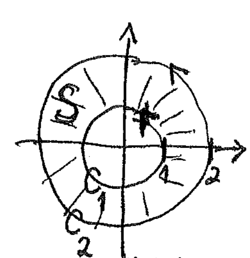
THM 6.10.4 (Stokes's surface Thm)

For a surface piece-with-boundary $S\ (\subset M) \subset \mathbb{R}^3$ and boundary curves $C_1, \dots, C_r$ and $\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$ a vector field

$$\int_S \text{curl}\,\vec{F} \cdot \vec{n}\, |d\vec{x}| = \int_S \Phi_{\text{curl}\,\vec{F}} = \int_S dW_{\vec{F}} = \int_{\partial S} W_{\vec{F}} = \sum_{i=1}^{r} \int_{C_i} W_{\vec{F}}$$

EXAMPLE: For S = annulus between $r=1$, $r=2$ in $\mathbb{R}^2$

and $\vec{F} = \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}$, let's check Green's Thm:



$[0,2\pi] \xrightarrow{\gamma_1} C_1$

$t \longmapsto \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$

$[0,2\pi] \xrightarrow{\gamma_2} C_2$

$t \longmapsto \begin{pmatrix} 2\cos t \\ 2\sin t \end{pmatrix}$

$$\sum_{i=1}^{2} \int_{C_i} (f\,dx + g\,dy) = +\int_{C_2} f\,dx + g\,dy - \int_{C_1} f\,dx + g\,dy = \int_0^{2\pi} \Big(\overset{-y}{-2\sin t} \cdot \overset{dx}{(-2\sin t)} + \overset{x}{2\cos t} \cdot \overset{dy}{2\cos t}\Big)\,dt$$

$$- \int_0^{2\pi} \big((-\sin t)(-\sin t) + \cos t \cdot \cos t\big)\,dt$$

$$= \int_0^{2\pi} 4\,dt - \int_0^{2\pi} 1\,dt = 3 \cdot 2\pi = 6\pi$$

(921) Meanwhile $\int_S \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx \wedge dy = \int_S \left(1-(-1)\right) dx \wedge dy = 2\int_S dx\, dy$

$= 2\,\text{area}(S) = 2\pi(2^2 - 1^2) = 6\pi$ ✓

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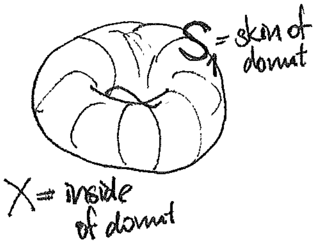
THM 6.10.6 (Gauss's Thm. or Divergence Thm.)

For X a piece-with-boundary in $\mathbb{R}^3$ and boundary surfaces $S_1, \dots, S_r$, a 2-form $\varphi = \Phi_{\bar F}$ for some $\bar F : \mathbb{R}^3 \to \mathbb{R}^3$ will have

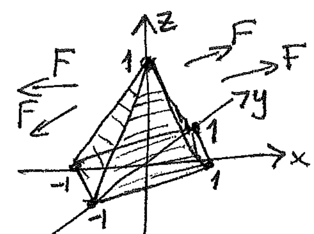
$$\int_X \operatorname{div} \bar F \, dx\, dy\, dz = \int_X d\varphi = \int_{\partial X} \varphi = \sum_{i=1}^{r} \underbrace{\int_{S_i}}_{\text{appropriately oriented}} \Phi_{\bar F} = \sum_{i=1}^{r} \int_{S_i} \bar F \cdot \bar n \, |d^2 \bar x|$$



EXAMPLE: Let $\bar F = \operatorname{curl}\begin{pmatrix} z^2 \\ z^3 \\ z^4 \end{pmatrix}$. What is $\int_S \Phi_{\bar F}$ for S = the four top slanted faces of this pyramid?

(A typical trick question involving Stokes-type theorems)

$$= \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{bmatrix} \times \begin{bmatrix} z^2 \\ z^3 \\ z^4 \end{bmatrix} = \begin{bmatrix} -3z^2 \\ -(-2z) \\ 0 \end{bmatrix}$$



Letting X = inside of pyramid

S' = bottom face of pyramid

then $\int_X \operatorname{div} \bar F \, dx\, dy\, dz = \int_S \Phi_{\bar F} + \int_{S'} \Phi_{\bar F}$ — with appropriate orientations

$$\| $$

$$0 = \int_X \underbrace{\operatorname{div}\operatorname{curl}\begin{pmatrix} z^2 \\ z^3 \\ z^4 \end{pmatrix}}_{0} dx\, dy\, dz$$

$$\Rightarrow \int_S \Phi_{\bar F} = -\int_{S'} \Phi_{\bar F} = 0$$

because $\bar F\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} -3z^2 \\ 2z \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ on the whole face S', where $z = 0$

See book's EXAMPLE 6.10.8 for a nice derivation of Archimedes's principle of buoyancy from Div. Thm!