(921)

Meanwhile $\int_S \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx \wedge dy = \int_S \left(1-(-1)\right) dx\wedge dy = 2\int_S dx\, dy$

$= 2\,\text{area}(S) = 2\pi(2^2-1^2) = 6\pi$ ✓

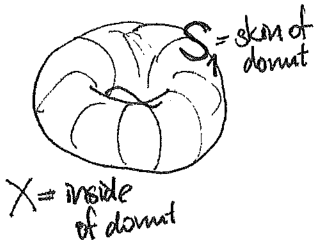
4/28/2017 >

THM 6.10.6 (Gauss's Thm. or Divergence Thm.)

For X a piece-with-boundary in $\mathbb{R}^3$ and boundary surfaces $S_1, \ldots, S_r$ a 2-form $\varphi = \Phi_{\vec F}$ for some $\vec F : \mathbb{R}^3 \to \mathbb{R}^3$ will have

$$\int_X \operatorname{div}\vec F\, dx\, dy\, dz = \int_X d\varphi = \int_{\partial X}\varphi = \sum_{i=1}^{r}\int_{S_i}\Phi_{\vec F} = \sum_{i=1}^{r}\int_{S_i}\vec F\cdot\vec n\, |d^2x|$$

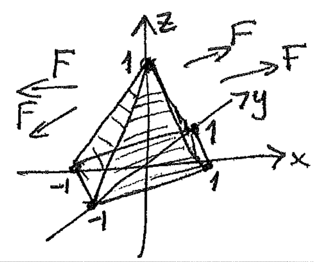
appropriately oriented



EXAMPLE: Let $\vec F = \operatorname{curl}\begin{pmatrix} z^2 \\ z^3 \\ z^4 \end{pmatrix}$. What is $\int_S \Phi_{\vec F}$ for S = the four top slanted faces of this pyramid?

(A typical trick question involving Stokes-type theorems)

$$= \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{bmatrix} \times \begin{bmatrix} z^2 \\ z^3 \\ z^4 \end{bmatrix} = \begin{bmatrix} -3z^2 \\ -(-2z) \\ 0 \end{bmatrix}$$



Letting X = inside of pyramid

S' = bottom face of pyramid

then $\int_X \operatorname{div}\vec F\, dx\, dy\, dz = \int_S \Phi_{\vec F} + \int_{S'}\Phi_{\vec F}$ with appropriate orientations

$\Vert$

$0 = \int_X \underbrace{\operatorname{div}\operatorname{curl}\begin{pmatrix} z^2 \\ z^3 \\ z^4\end{pmatrix}}_{0} dx\, dy\, dz$

$\Rightarrow \int_S \Phi_{\vec F} = -\int_{S'}\Phi_{\vec F} = 0$

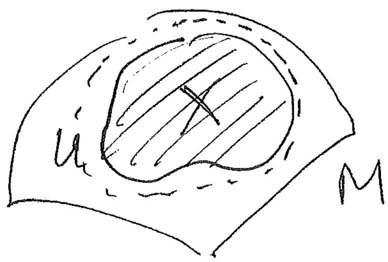
because $\vec F\begin{pmatrix} x \\ y \\ z\end{pmatrix} = \begin{bmatrix} -3z^2 \\ 2z \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ on the whole face S', where $z=0$

See book's EXAMPLE 6.10.8 for a nice derivation of Archimedes's principle of buoyancy from Div. Thm!

§6.9 General Stokes's Thm (+ sketch informal proof)

THM 6.9.2 $\quad X^{\text{piece-with-boundary}} \subset M^{k\text{-dim'l manifold}}$ and $\varphi \in A^{k-1}(\mathcal{U})$ with coefficients in $C^2(\mathcal{U})$

$$\begin{array}{ccc} X & \subset & M \\ \cap & & \cap \\ \mathcal{U}^{\text{open}} & \subset & \mathbb{R}^n \end{array}$$



$$\Rightarrow \int_{\partial X} \varphi = \int_X d\varphi$$

given boundary orientation induced from X

Informal sketch proof:

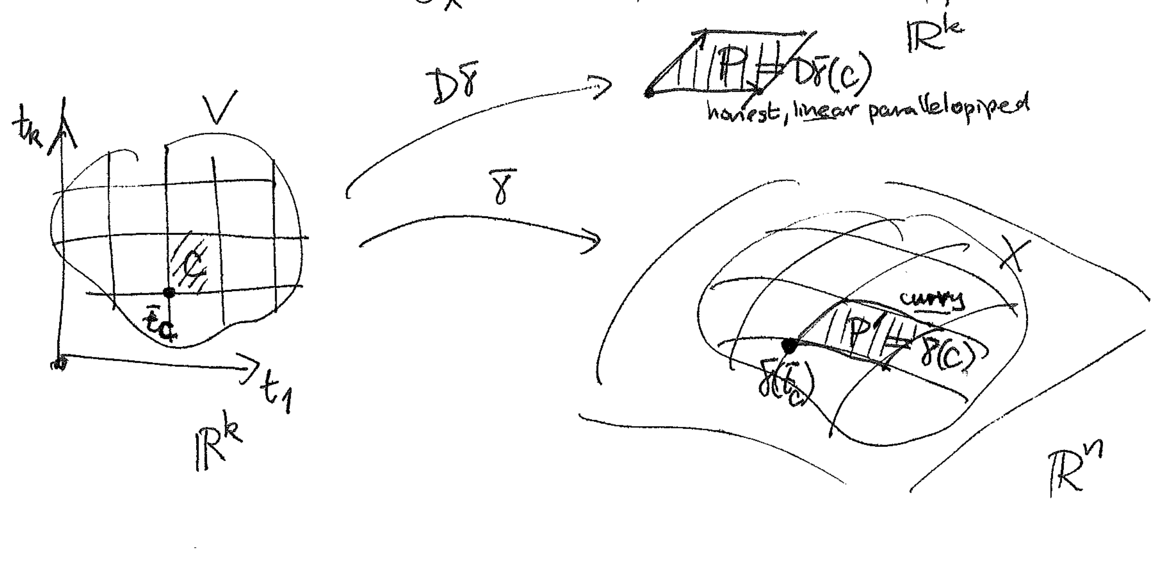
A key point is that if we rephrase our definition of $d\varphi$, then it shows we have made Stokes's Thm true infinitesimally (by defn!) for small parallelepipeds $P = P_{\bar{x}}(h\bar{v}_1, \ldots, h\bar{v}_k)$ with tiny h:

$$d\varphi(P_{\bar{x}}(\bar{v}_1, \ldots, \bar{v}_k)) \overset{\text{DEF'N}}{:=} \lim_{h \to 0} \frac{1}{h^k} \int_{\partial P_{\bar{x}}(h\bar{v}_1, \ldots, h\bar{v}_k)} \varphi = \lim_{h \to 0} \frac{1}{h^k} \int_{\partial P} \varphi$$

$$= \frac{1}{h^k} d\varphi(P_{\bar{x}}(h\bar{v}_1, \ldots, h\bar{v}_k)) = \frac{1}{h^k} d\varphi(P)$$

i.e. $\boxed{d\varphi(P) \approx \int_{\partial P} \varphi}$

Now if we compute $\int_X d\varphi$ via a parametrization $V^{\text{open}} \xrightarrow{\bar{\gamma}} X$, $V \subset \mathbb{R}^k$

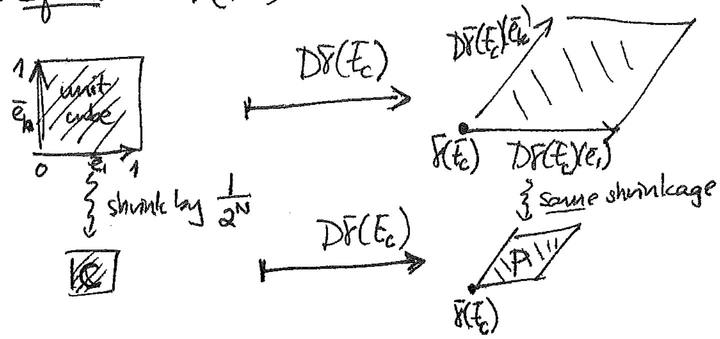


(123)

$$\int_X d\varphi := \int_V d\varphi\Big(P_{\gamma(E)}\big(\underbrace{D\gamma(E)(\bar e_1),\dots,D\gamma(E)(\bar e_k)}_{D\gamma(E)\text{ as a matrix of column vectors}}\big)\Big)\,|d^kE|$$

$$= \lim_{N\to\infty} \sum_{\substack{C\in\mathcal{D}_N(\mathbb{R}^k):\\ C\subset V}} \underbrace{d\varphi\big(P_{\gamma(E_C)}(D\gamma(E_C))\big)\,\mathrm{vol}_k(C)}_{\text{this equals } d\varphi(P) \text{ where } P := D\gamma(E_C)(C):}$$

use Riemann sums with E_C as sample point in C



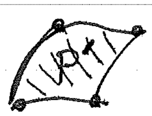
$$= \lim_{N\to\infty} \sum_C d\varphi(P)$$

Boxed fact from before

$$\approx \lim_{N\to\infty} \sum_C \int_{\partial P} \varphi$$

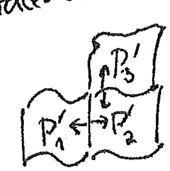
P' is close to P for C small

$$\approx \lim_{N\to\infty} \sum_C \int_{\partial P'} \varphi$$

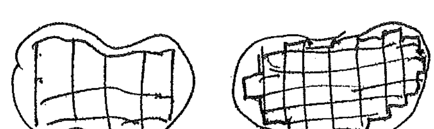


$$= \int_{\partial(\bigcup_C P')} \varphi \approx \int_{\partial X} \varphi \; .$$

Because outward normals for adjacent P' are opposite, so interior faces of P' cancel!



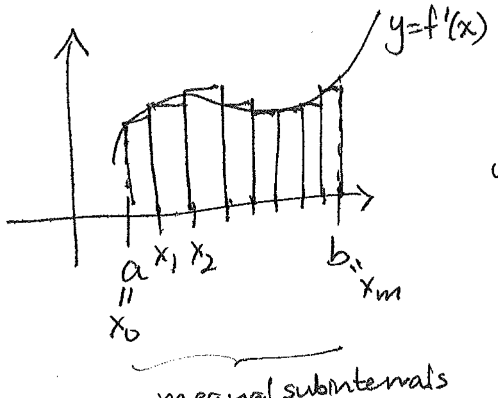
the boundary of $\bigcup_C P'$ approximates ∂X better and better in the limit



what is slippery here is that the approximations "$\approx$" have small errors, and the errors have to be summed over sums that have more and more terms, so one might worry that they don't stay arbitrarily small!

5/1/2017 REASSURANCE 1:

The book sketches its 2^{nd} proof of Fund'l Thm. of Calculus, and bounds the errors...



with $f \in C^2(\mathcal{U})$ for some $\mathcal{U} \overset{\text{open}}{\supset} [a,b]$

Informally first, for m large

$$\underbrace{\int_a^b f'(x)\,dx}_{A} \approx \underbrace{\sum_{i=0}^{m-1} f'(x_i)(x_{i+1}-x_i)}_{B_m} \approx \underbrace{\sum_{i=0}^{m}\left(f(x_{i+1})-f(x_i)\right)}_{C_m} \overset{\text{telescoping sum}}{=} f(b)-f(a)$$

because Mean Value Thm. gives some $c_i \in (x_i, x_{i+1})$ with $f(x_{i+1})-f(x_i) = f'(c_i)(x_{i+1}-x_i)$, and $f \in C^1 \Rightarrow f'(c_i) \approx f'(x_i)$

To bound the errors, given $\epsilon > 0$, suffices to show $\exists M$ such that $\forall m \geq M$

$$|A - B_m| < \epsilon$$

and $|B_m - C_m| < \epsilon$.

For $|A - B_m| < \epsilon$, this is just Riemann integrability of f', which is continuous on $[a,b]$, since $f \in C^2(\mathcal{U})$ for $[a,b] \subset \mathcal{U}$.

For $|B_m - C_m| < \epsilon$, use uniform continuity of f' on $[a,b]$ (f' is continuous, $[a,b]$ is compact) to choose M so $m \geq M$ implies $|f'(c_i) - f'(x_i)| \leq \epsilon$

and then

$$|B_m - C_m| = \left|\sum_{i=0}^{m-1} f'(x_i)(x_{i+1}-x_i) - \underbrace{(f(x_{i+1})-f(x_i))}_{=f'(c_i)(x_{i+1}-x_i)}\right| = \left|\sum_{i=0}^{m-1}(f'(x_i)-f'(c_i))(x_{i+1}-x_i)\right|$$
$$\leq \sum_{i=0}^{m-1}|f'(x_i)-f'(c_i)|\,|x_{i+1}-x_i|$$
$$\leq \epsilon \sum_{i=0}^{m-1}|x_{i+1}-x_i| \leq \epsilon(b-a).$$