(84)
2/27/2017

§4.5 Fubini's Theorem

How to actually calculate $\int_{\mathbb{R}^n} f(\bar{x})\,|d^n\bar{x}|$ avoiding Riemann sums?

We know Fund'l Thm. of Calc for $n=1$: If $f(x) = \underset{\text{continuous}}{\frac{d}{dx}} F(x)$, $\int_a^b f(x)\,dx = F(b) - F(a)$

and if no antiderivative $F(x)$ is available, there are numerical approximation methods (§4.6).

The F.T.C. exact method for $n \geq 2$ relies on...

(Fubini's) Thm 4.5.10: If $\mathbb{R}^{n+m} \xrightarrow{f} \mathbb{R}$ is integrable

and for each $\bar{x} \in \mathbb{R}^n$, the function $\mathbb{R}^m \xrightarrow{f_{\bar{x}}} \mathbb{R}$ is integrable,
$\bar{y} \longmapsto f_{\bar{x}}(\bar{y}) = f(\bar{x},\bar{y})$

$$\text{then } \int_{\mathbb{R}^{n+m}} f(\bar{x},\bar{y})\,|d^n\bar{x}\, d^m\bar{y}| = \int_{\mathbb{R}^n} \underbrace{\left(\int_{\mathbb{R}^m} f_{\bar{x}}(\bar{y})\,|d^m\bar{y}| \right)}\,|d^n\bar{x}|$$

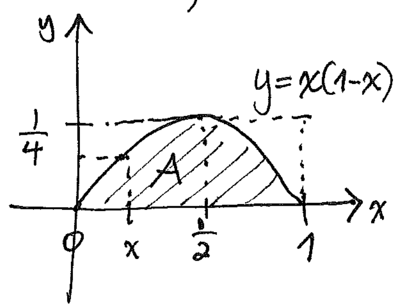
called an iterated integral

... and this thing is an integrable function $\mathbb{R}^n \to \mathbb{R}$, $x \longmapsto \int_{\mathbb{R}^m} f_{\bar{x}}(\bar{y})\,|d^m\bar{y}|$

Let's see how to use it before proving it.

EXAMPLES:

(1) For this $A \subset \mathbb{R}^2$,



what are $\int_A |dx\,dy|$

(i) area of A?

(ii) center of mass of A

$$= \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \overset{\text{DEF'N}}{\underset{4.21}{:=}} \frac{1}{\text{area of } A} \begin{pmatrix} \int_A x\,|dx\,dy| \\ \int_A y\,|dx\,dy| \end{pmatrix}$$

$$= \begin{pmatrix} E(x) \\ E(y) \end{pmatrix}$$ (expected value) where $\binom{x}{y}$ has prob. density uniform on A, i.e.

$$\mu(x,y) = \begin{cases} 0 & \text{if } \binom{x}{y} \notin A \\ \frac{1}{\text{area of } A} & \text{if } \binom{x}{y} \in A \end{cases}$$

For (i),

$$\text{area of } A = \int_A |dx\,dy| = \int_{\mathbb{R}^2} 1_A(x,y)\,|dx\,dy|$$

$$= \int_{x=0}^{x=1} \left(\int_{\mathbb{R}^1} 1_A(x,y)\,|dy| \right) dx$$

$$= \int_{x=0}^{x=1} \left(\int_{y=0}^{y=x(1-x)} dy \right) dx = \int_{x=0}^{x=1} [\underbrace{x(1-x)}_{x-x^2} - 0]\,dx = \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_{x=0}^{x=1} = \frac{1}{2} - \frac{1}{3} = \frac{1}{6} \quad \left(\overset{\checkmark}{<} \frac{1}{4}\right)$$

(55)

For (ii), center of mass has

$$x_0 = \frac{1}{1/6}\int_{\mathbb{R}^2} x\cdot 1_A(x,y)\,|dx\,dy| = 6\int_{x=0}^{x=1}\left(\int_{y=0}^{y=x(1-x)} x \; dy\right)dx$$

$$= 6\int_{x=0}^{x=1} x\left(\int_{y=0}^{x(1-x)} dy\right)dx = 6\int_{x=0}^{x=1}\underbrace{x\cdot x(1-x)}_{x^2-x^3}\,dx = 6\left[\frac{x^3}{3}-\frac{x^4}{4}\right]_{x=0}^{x=1} = 6\left[\frac{1}{3}-\frac{1}{4}\right] = \frac{6}{12} = \frac{1}{2} \checkmark$$

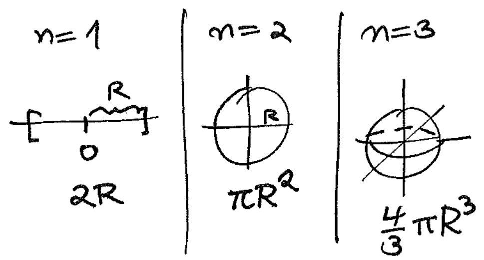
(why?)

$$y_0 = \frac{1}{1/6}\int_{x=0}^{x=1}\left(\int_{y=0}^{x(1-x)} y\,dy\right)dx = 6\int_{x=0}^{x=1}\left[\frac{y^2}{2}\right]_{y=0}^{y=x(1-x)}dx = \frac{6}{2}\int_{x=0}^{x=1}\underbrace{x^2(1-x)^2}_{x^2-2x^3+x^4}\,dx$$

$$= 3\left[\frac{x^3}{3}-\frac{2x^4}{4}+\frac{x^5}{5}\right]_{x=0}^{x=1} = 3\left[\frac{1}{3}-\frac{1}{2}+\frac{1}{5}\right] = 3\left[\frac{10-15+6}{30}\right] = \frac{3}{30} = \frac{1}{10}$$

$\left(\le \frac{1}{8}\right)$ ✓

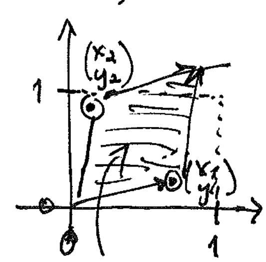
(2) See book's iterative calculation of vol_n (ball of radius R in $\mathbb{R}^n$) EXAMPLE 4.5.7,



using Fubini and slices having vol_{n-1} known by induction

(3) See EXAMPLE 4.5.9: Expected value of area of parallelogram shown, where $\binom{x_1}{y_1}, \binom{x_2}{y_2}$ have x_1, x_2, y_1, y_2 chosen uniformly on $[0,1]$?

$$\int_0^1\int_0^1\int_0^1\int_0^1 |x_1y_2 - x_2y_1|\,dy_2\,dx_2\,dy_1\,dx_1$$



area $= \left|\det\begin{bmatrix} x_1 & x_2 \\ y_1 & y_2\end{bmatrix}\right| = |x_1y_2 - x_2y_1|$

The trick bit is how to get rid of absolute values; see book's discussion.

Fubini as we stated it actually follows from something stronger (in Appendix A.17):

<u>THM A.17.2:</u> $\mathbb{R}^{n+m} \xrightarrow{f} \mathbb{R}$ integrable $\Rightarrow$ $\mathbb{R}^m \xrightarrow{f_{\bar{x}}} \mathbb{R}$ has $\mathcal{U}(f_{\bar{x}}), \mathcal{L}(f_{\bar{x}})$ integrable,

$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} \longmapsto f(\bar{x},\bar{y})$ $\qquad$ $\bar{y} \longmapsto f(\bar{x},\bar{y})$

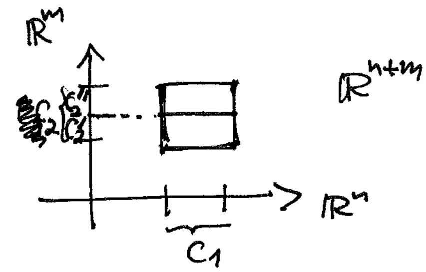
and
$$\int_{\mathbb{R}^{n+m}} f(\bar{x},\bar{y})\,|d^n\bar{x}|\,|d^m\bar{y}| = \int_{\mathbb{R}^n} \mathcal{U}(f_{\bar{x}})\,|d^n\bar{x}| = \int_{\mathbb{R}^n} \mathcal{L}(f_{\bar{x}})\,|d^n\bar{x}|$$
(These would have been the same if $f_{\bar{x}}$ was assumed integrable.)

proof: The strategy will be to trap things between $U_N(f) \geq L_N(f) \ldots$

$$\begin{array}{ccc} U_N(f) & \geq & L_N(f) \\ \downarrow & {\scriptstyle N\to\infty} & \downarrow \\ \mathcal{U}(f) & = & \mathcal{L}(f) \end{array}$$
f integrable

If we fix N, then ~~claim~~ that for any $N' \geq N$,

$U_N(f) \geq U_N(\mathcal{U}_{N'}(f_{\bar{x}}))$ because



$$U_N(f) = \sum_{C \in \mathcal{D}_N(\mathbb{R}^{n+m})} M_C(f)\,\mathrm{vol}_{n+m}(C)$$

$$= \sum_{C_1 \in \mathcal{D}_N(\mathbb{R}^n)} \sum_{C_2 \in \mathcal{D}_N(\mathbb{R}^m)} M_{C_1\times C_2}(f)\,\mathrm{vol}_n(C_1)\,\mathrm{vol}_m(C_2)$$

Think about it!

$$\geq \sum_{C_1 \in \mathcal{D}_N(\mathbb{R}^n)} \sum_{C_2' \in \mathcal{D}_{N'}(\mathbb{R}^m)} \overbrace{M_{C_1\times C_2'}(f)}^{= M_{C_1}(M_{C_2}(f_{\bar{x}}))}\,\mathrm{vol}_n(C_1)\,\mathrm{vol}_m(C_2')$$

$$= \sum_{C_1 \in \mathcal{D}_N(\mathbb{R}^n)} M_{C_1}\Bigg(\underbrace{\sum_{C_2 \in \mathcal{D}_{N'}(\mathbb{R}^m)} M_{C_2}(f_{\bar{x}})\,\mathrm{vol}_m(C_2')}_{U_{N'}(f_{\bar{x}})}\Bigg)\,\mathrm{vol}_n(C_1) = U_N(U_{N'}(f_{\bar{x}}))$$

Then we have
$$U_N(f) \geq U_N(U_{N'}(f_{\bar{x}})) \geq L_N(L_{N'}(f_{\bar{x}})) \geq L_N(f)$$

$\boxed{f \geq g \Rightarrow U_N(f) \geq U_N(g) \text{ and } U_N(f) \geq L_N(f)}$ proven same way as for U's

$\downarrow$ take $\lim_{N'\to\infty}$ with N still fixed

$$\begin{array}{ccccccc} & & & L_N(\mathcal{U}(f_{\bar{x}})) & & & \\ & & \rotatebox{45}{$\leq$} & & \rotatebox{-45}{$\geq$} & & \\ U_N(f) \geq U_N(\mathcal{U}(f_{\bar{x}})) & & & \geq & & L_N(\mathcal{L}(f_{\bar{x}})) \geq L_N(f) & \\ & & \rotatebox{-45}{$\geq$} & & \rotatebox{45}{$\geq$} & & \\ & & & U_N(\mathcal{L}(f_{\bar{x}})) & & & \end{array}$$

$\downarrow$ take $\lim_{N\to\infty}$

$$\mathcal{U}(f) = \mathcal{U}(\mathcal{U}(f_{\bar{x}})) = \mathcal{L}(\mathcal{U}(f_{\bar{x}})) = \mathcal{U}(\mathcal{L}(f_{\bar{x}})) = \mathcal{L}(\mathcal{L}(f_{\bar{x}})) = L_N(f) \quad \Big(= \int_{\mathbb{R}^{n+m}} f(\bar{x},\bar{y})\,|d^n\bar{x}\,d^m\bar{y}|\Big)$$ ∎