(14) We also saw how for inverse functions

$$\begin{array}{ccc} U^{\text{open}} & \underset{\bar g = \bar f^{-1}}{\overset{\bar f}{\rightleftarrows}} & V^{\text{open}} \\ \cap & & \cap \\ \mathbb{R}^n & & \mathbb{R}^n \end{array}$$

one can similarly use
$$[D\bar g(\bar y)] = [D\bar f(\bar x)]^{-1} = [D\bar f(\bar g(\bar y))]^{-1} = \frac{1}{\det\left[\frac{\partial f_i}{\partial x_j}(\bar g(\bar y))\right]} \operatorname{adj}\left[\frac{\partial f_i}{\partial x_j}(\bar g(\bar y))\right]^T$$

to show $\bar f \in C^k \Rightarrow \bar g \in C^k$ again via induction on k.

Similarly, for implicit functions $\bar F\begin{pmatrix}\bar x \\ g(\bar x)\end{pmatrix} = \bar 0$

the relation $D\bar g(\bar a) = -\left[D\bar F(\bar c)\big|_{\substack{\text{pivot}\\\text{vars}}}\right]^{-1}\left[D\bar F(\bar c)\big|_{\substack{\text{nonpivot}\\\text{variables}}}\right]$

can be used to show $\bar F \in C^k \Rightarrow \bar g \in C^k$ via induction on k.

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But then, one still needs to show that if $\underset{f(\bar a + \bar h)}{\overset{\shortparallel}{f(\bar x)}} = P_{f,\bar a}(\bar a + \bar h) + o(|\bar h|^k)$

and $g(\bar x) = P_{g,\bar a}(\bar a + \bar h) + o(|\bar h|^k)$

then $f(\bar x) + g(\bar x) = P_{f,\bar a}(\bar a + \bar h) + P_{g,\bar a}(\bar a + \bar h) + o(|\bar h|^k)$

easy since $o(|\bar h|^k) + o(|\bar h|^k) \subset o(|\bar h|^k)$

$$\begin{aligned} f(\bar x) g(\bar x) &= (P_{f,\bar a} + o(|\bar h|^k))(P_{g,\bar a} + o(|\bar h|^k)) \\ &= P_{f,\bar a} P_{g,\bar a} + P_{f,\bar a}\, o(|\bar h|^k) + P_{g,\bar a}\, o(|\bar h|^k) + \underbrace{o(|\bar h|^k)\, o(|\bar h|^k)}_{\text{in } o(|\bar h|^{2k})} \\ &= \begin{pmatrix}\deg \le k \text{ terms} \\ \text{of } P_{f,\bar a} P_{g,\bar a}\end{pmatrix} + \underbrace{o(|\bar h|^k) + (P_{f,\bar a} + P_{g,\bar a})\, o(|\bar h|^k)}_{\text{all in } o(|\bar h|^k)\text{; not hard, see appendix A.11}} \subset o(|\bar h|^k) \end{aligned}$$

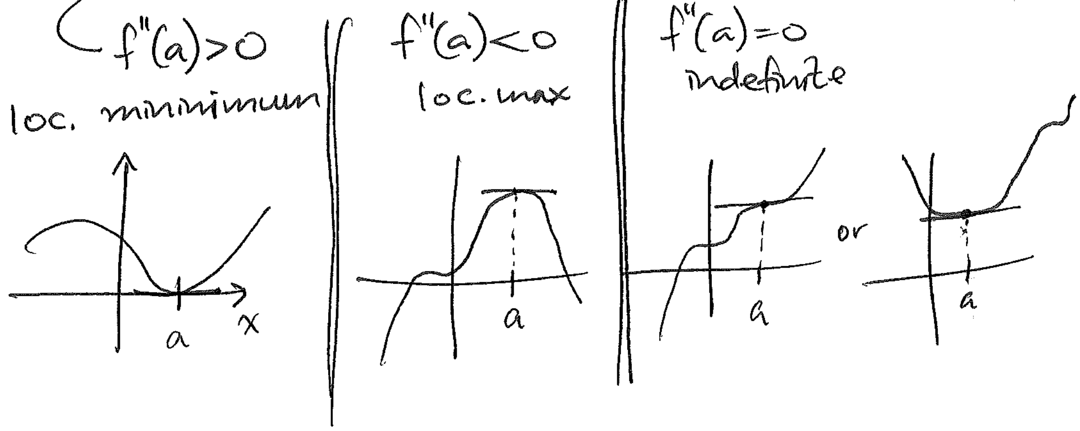
... etc for $(g \circ f)(\bar x)$ and for $F\begin{pmatrix}\bar x \\ \bar g(\bar x)\end{pmatrix} = 0$; see appendix A.11.

§3.5 Quadratic forms

How did one search for extrema (maxes/mins) of $f(x)$ in 1-variable?

Found local extrema if f differentiable where $f'(a) = 0$, and if f was twice differentiable, classified them via 2nd derivative test:

$$f(x) = f(a) + \underbrace{\cancel{f'(a)(x-a)}}_{=0} + \frac{f''(a)}{2}(x-a)^2 + o((x-a)^2)$$



We'll proceed similarly in §3.6 for $\mathbb{R}^n \xrightarrow{f} \mathbb{R}$, analyzing the quadratic part of $P^2_{f,\bar{a}}(x)$. As usual, WLOG $\bar{a} = \bar{0}$, and then it looks like one of these things:

DEF'N 3.5.1: A quadratic form is $\mathbb{R}^n \xrightarrow{Q} \mathbb{R}$

$$\bar{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \longmapsto Q(\bar{x}) = \overbrace{\sum_{i=1}^{n} a_i x_i^2 + \sum_{1 \le i < j \le n} b_{ij} x_i x_j}^{\text{all quadratic terms}}$$

for some $a_i, b_{ij} \in \mathbb{R}$

EXAMPLES: ① $Q\begin{pmatrix} x \\ y \\ z \end{pmatrix} = (x+2y-3z)^2$

② $Q\begin{pmatrix} x \\ y \end{pmatrix} = x^2+5y^2$

③ $Q\begin{pmatrix} x \\ y \end{pmatrix} = xy$

④ For any symmetric $n\times n$ matrix $A = A^T$
$$\overset{\shortparallel}{(a_{ij})_{\substack{i=1,\dots,n\\ j=1,\dots,n}}} \text{ with } a_{ij}=a_{ji},$$

there is an associated quadratic form
$$Q_A(\bar{x}) \underset{\text{DEF'N}}{:=} \bar{x}^T A \bar{x} = \sum_{i=1}^{n} a_{ii}x_i^2 + \sum_{1\le i<j\le n} 2a_{ij}x_ix_j$$

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e.g. $n=2$
$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22}\end{bmatrix} \text{ has } Q_A\begin{pmatrix}x\\y\end{pmatrix} = [x\ y]\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix}\begin{bmatrix}x\\y\end{bmatrix}$$
$$= [x\ y]\begin{bmatrix} a_{11}x+a_{12}y \\ a_{12}x+a_{22}y\end{bmatrix}$$
$$= a_{11}x^2 + a_{12}xy + a_{12}xy + a_{22}y^2$$
$$= a_{11}x^2 + a_{22}y^2 + 2a_{12}xy$$

And, in fact, every quadratic form $Q(\bar{x}) = \sum_{i=1}^{n} a_ix_i^2 + \sum_{1\le i<j\le n} b_{ij}x_ix_j$

has $Q = Q_A$ for some symmetric A, namely let $a_{ii} := a_i$, $a_{ij} = \frac{1}{2}b_{ij}$, i.e.
$$A = \begin{bmatrix} a_1 & \frac{b_{12}}{2} & \frac{b_{13}}{2} & \dots \\ \frac{b_{12}}{2} & a_2 & & \\ \frac{b_{13}}{2} & & \ddots & \\ \vdots & & & a_n \end{bmatrix}$$

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⑤ The quadratic Taylor polynomial $P^2_{f,\bar{0}}(\bar{x})$ for $\mathbb{R}^n \xrightarrow{f} \mathbb{R}$

looks like
$$P^2_{f,\bar{0}}(\bar{x}) = f(\bar{0}) + \sum_{i=1}^{n}\frac{\partial f}{\partial x_i}(\bar{0})\cdot x_i + \sum_{i=1}^{n}\frac{\frac{\partial^2 f}{\partial x_i^2}(\bar{0})}{2!}\cdot x_i^2 + \sum_{1\le i<j\le n}\frac{\partial^2 f}{\partial x_i\partial x_j}(0)x_ix_j$$
$$= f(\bar{0}) + \underset{\substack{\text{gradient of } f\\ \text{at } \bar{0}}}{\nabla f(\bar{0})^T}\bar{x} + \frac{1}{2}\overbrace{Q_H(\bar{x})}^{\bar{x}^T H\bar{x}} \text{ where}$$

$$H = \left[\frac{\partial^2 f}{\partial x_i x_j}(\bar{0})\right]_{\substack{i=1,\dots,n\\ j=1,\dots,n}} = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2}(\bar{0}) & \frac{\partial^2 f}{\partial x_1\partial x_2}(\bar{0}) & \dots \\ \frac{\partial^2 f}{\partial x_1\partial x_2}(\bar{0}) & \ddots & \\ \vdots & & \frac{\partial^2 f}{\partial x_n^2}(\bar{0}) \end{bmatrix}$$

is called the Hessian matrix of f at $\bar{x}=\bar{0}$.