(6) The point of the $\frac{1}{m!}$ was to make $P^k_{f,a}(x)$ have same derivatives at $x=a$ as $f(x)$ up through k^{th} derivative.

In several variables, say $\mathbb{R}^n \xrightarrow{f} \mathbb{R}$

$$\bar{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \longmapsto f(\bar{x})$$

has many higher partial derivatives. How to index them?

EXAMPLE: $\mathbb{R}^2 \xrightarrow{f} \mathbb{R}$

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \sin(x^2+y)$$

$\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$

$$\frac{\partial f}{\partial x} = 2x\cos(x^2+y) \qquad \frac{\partial f}{\partial y} = \cos(x^2+y)$$

$\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$

$$\frac{\partial^2 f}{\partial x^2} = 2\cos(x^2+y) - 4x^2\sin(x^2+y)$$

$$\frac{\partial^2 f}{\partial y \partial x} = -2x\sin(x^2+y) = \frac{\partial^2 f}{\partial x \partial y}$$

SAME!

$$\frac{\partial^2 f}{\partial y^2} = -\sin(x^2+y)$$

$\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$

$$\frac{\partial^3 f}{\partial x^3} = -8x^3\cos(x^2+y) - 12x\sin(x^2+y)$$

$$\frac{\partial^3 f}{\partial x^2 \partial y} = \frac{\partial^3 f}{\partial x \partial y \partial x} = \frac{\partial^3 f}{\partial y \partial x^2} = -4x^2\cos(x^2+y)$$

$$\frac{\partial^3 f}{\partial x \partial y^2} = \frac{\partial^3 f}{\partial y \partial x \partial y} = \frac{\partial^3 f}{\partial y^2 \partial x} = -2x\cos(x^2+y)$$

$$\frac{\partial^3 f}{\partial y^3} = -\cos(x^2+y)$$

THM 3.3.9 + COR 3.3.11: If $\mathbb{R}^n \xrightarrow{f} \mathbb{R}$ has each $\frac{\partial f}{\partial x_i}$ differentiable, then $\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}$ $\forall i,j$

If it has each $(k-1)^{st}$ order partial derivative differentiable, then each k^{th} order partial derivative is independent of order of $\frac{\partial}{\partial x_i}$'s.

missed lectures 1/23, 1/25; Collazos lectured 1/27

1/30/2017

Assuming this for the moment, define $D_{\underbrace{(i_1, i_2, \ldots, i_n)}_{I}} f := \frac{\partial^k f}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}} = D_I f$ where $k := i_1 + \cdots + i_n$

(actually, let's not do the proof in lecture; see proof in book Appendix A.9, or Brubaker's Lecture 31 showing $\frac{\partial^2 f}{\partial x_j \partial x_i} = \lim_{t \to 0} \frac{1}{t^2}\left(f(\bar{a}+t\bar{e}_i+t\bar{e}_j) - f(\bar{a}+t\bar{e}_j) - f(\bar{a}+t\bar{e}_i) + f(\bar{a})\right)$, which is symmetric in i & j)

(7) Note that $D_I f = \frac{\partial^k f}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}}$ for $I = (i_1, \ldots, i_n)$ with $i_1 + \cdots + i_n = k$

has the property that for any monomial $\bar{x}^J \overset{\text{DEF'N}}{:=} x_1^{j_1} \cdots x_n^{j_n}$,

one has $$D_I \bar{x}^J \Big|_{x_1 = x_2 = \cdots = x_n = 0} = \begin{cases} 0 & \text{if } I \neq J \\ \underbrace{i_1!\, i_2! \cdots i_n!}_{\text{DEF'N: Call this } I!} & \text{if } I = J \\ & \phantom{\text{if }} \overset{\parallel}{(i_1, \ldots, i_n)} \end{cases}$$

simply because $$\frac{\partial^m}{\partial x_i^m} \cdot x_i^l \Big|_{x_i = 0} = \begin{cases} 0 & \text{if } m \neq l \\ m! & \text{if } m = l \end{cases}$$

EXAMPLES:
$$\frac{\partial^5}{\partial x^3 \partial y^2}(x^4 y^{10}) \Big|_{x=y=0} = 4 \cdot 3 \cdot 2 \cdot x^1 \cdot 10 \cdot 9 y^8 \Big|_{x=y=0} = 0$$

$$\frac{\partial^5}{\partial x^3 \partial y^2}(x^4 y^1) \Big|_{x=y=0} = 0$$

$$\frac{\partial^5}{\partial x^3 \partial y^2}(xy) \Big|_{x=y=0} = 0$$

$$\frac{\partial^5}{\partial x^3 \partial y^2}(x^3 y^2) \Big|_{x=y=0} = 3 \cdot 2 \cdot 1 \cdot 2 \cdot 1 = 3!\,2!$$

DEF'N 3.3.14: For $\bar{a} \in \mathcal{U} \xrightarrow{f} \mathbb{R}$, $\mathcal{U} \subset \mathbb{R}^n$, and f having all k^{th} order partials $D_I f(\bar{a})$ with $i_1 + \cdots + i_n \leq k$ defined (and indep. of order),

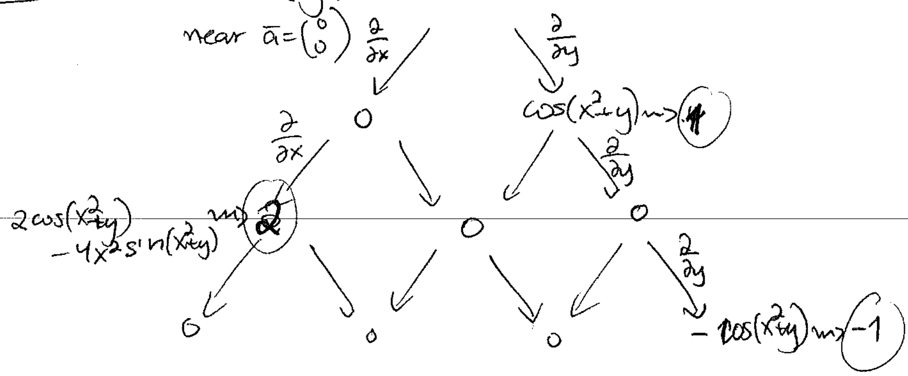
define the k^{th} degree Taylor polynomial for f near $\bar{x} = \bar{a}$

to be $$P^k_{f,\bar{a}}(\bar{a} + \bar{h}) := \sum_{\substack{I = (i_1, \ldots, i_n) \\ i_1 + \cdots + i_n \leq k}} \frac{D_I f(\bar{a})}{I!} \bar{h}^I$$

$\bar{x} = \bar{a} + \bar{h}$
$\bar{h} = \bar{x} - \bar{a}$
or equivalently

$$P^k_{f,\bar{a}}(\bar{x}) := \sum_{\substack{I \\ \sum i_j \leq k}} \frac{D_I f(\bar{a})}{I!} (\bar{x} - \bar{a})^I$$

EXAMPLE: $f\begin{pmatrix}x\\y\end{pmatrix} = \sin(x^2+y)$ $\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix} \rightsquigarrow 0$



$$\Rightarrow P^3_{f,\begin{pmatrix}0\\0\end{pmatrix}}\begin{pmatrix}x\\y\end{pmatrix} = \frac{1\cdot y}{1!} + \frac{2x^2}{2!} + \frac{(-1)y^3}{3!} = x^2+y-\frac{y^3}{3!}$$

THM 3.3.7: In this setting, $P^k_{f,\bar{a}}(\bar{x})$ is the unique polynomial of deg $\le k$ ...

(i) .... ~~such that~~ with same partial derivatives at $\bar{x}=\bar{a}$ of f up to order k, i.e. $D_I f(\bar{a}) = D_I P^k_{f,\bar{a}}(\bar{a})$ $\forall I=(i_1,\dots,i_n)$ with $\sum i_j \le k$.

(ii) ... with $$\lim_{\bar{h}\to 0} \frac{f(\bar{a}+\bar{h}) - P^k_{f,\bar{a}}(\bar{a}+\bar{h})}{|\bar{h}|^k} = 0$$

RMK: It's convenient to rephrase (ii) above in terms of Landau's "little oh" notation:

DEF'N 3.4.1: Say $p(\bar{x}), g(\bar{x})$ have $p = o(g)$ (as $\bar{x}\to\bar{b}$) "p is little oh of g"

if $$\lim_{\bar{x}\to\bar{b}} \frac{p(\bar{x})}{g(\bar{x})} = 0$$

Then (ii) above says $f(\bar{a}+\bar{h}) - P^k_{f,\bar{a}}(\bar{a}+\bar{h}) = o(|\bar{h}|^k)$ as $\bar{h}\to\bar{0}$.

RMK: If one assumes more, namely $f\in C^{k+1}(U)$ for some $U^{\text{open}} \supseteq [\bar{a},\bar{a}+\bar{h}]$ then one can be more precise (THM A12.7) in (ii):

$$f(\bar{a}+\bar{h}) - P^k_{f,\bar{a}}(\bar{a}+\bar{h}) \le C\left(\sum_{i=1}^n |h_i|\right)^{k+1}$$

where $$C := \sup_{\substack{I=(i_1,\dots,i_n)\\ i_1+\dots+i_n = k+1\\ \bar{c}\in[\bar{a},\bar{a}+\bar{h}]}} |D_I f(\bar{c})|$$

proof of THM 3.3.7:

WLOG one can assume $\bar{a}=\bar{0}$; replace $\bar{x}=\bar{a}+\bar{h}$
$\bar{h}=\bar{x}-\bar{a}$.

(i): Any polynomial $P(\bar{x})$ of degree $\leq k$ has the form

$$P(\bar{x}) = \sum_{\substack{J:\\ j_1+\dots+j_n\leq k}} c_J \bar{x}^J \qquad \text{for some } c_J \in \mathbb{R}$$

and hence $D_I P(\bar{0}) = D_I\left(\sum_J c_J \bar{x}^J\right)\Big|_{\bar{x}=\bar{0}}$

for $I=(i_1,\dots,i_n)$, $i_1+\dots+i_n\leq k$

$$= \sum_J c_J \left[D_I \bar{x}^J\right]_{\bar{x}=\bar{0}}$$

$$= c_I \cdot I!$$

Hence $D_I P(\bar{0}) = D_I f(\bar{0}) \quad \forall I \text{ with } \sum i_j \leq k$

$\iff c_I \cdot I! = D_I f(\bar{0})$

$\iff c_I = \dfrac{D_I f(\bar{0})}{I!}$, i.e. $P = P^k_{f\bar{a}}$

(ii): Note that $g(\bar{x}) := f(\bar{x}) - P^k_{f,\bar{a}}(\bar{x})$ has $D_I g(\bar{a})=0 \quad \forall i_1+\dots+i_n\leq k$,

so part of (ii) follows from this **lemma:**

LEMMA: (PROP 3.3.18) If $D_I g(\bar{a})=0 \quad \forall i_1+\dots+i_n\leq k$ (including $g(\bar{a})=0$)

then $\displaystyle\lim_{\bar{h}\to\bar{0}} \frac{g(\bar{a}+\bar{h})}{|\bar{h}|^k} = 0$.

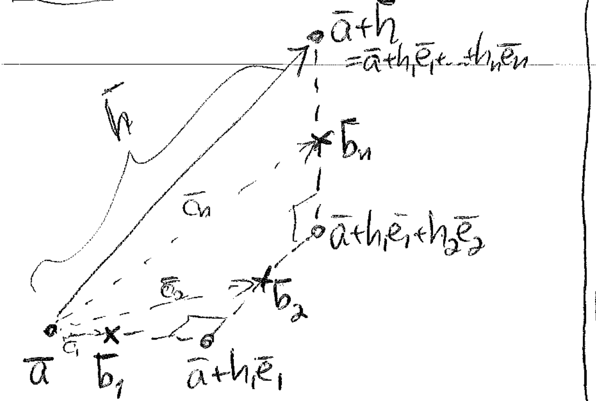
proof: Induct on k, with base case $k=0$ following from $g(\bar{a})=0$ and continuity of g:

$$\lim_{\bar{h}\to\bar{0}} \frac{g(\bar{a}+\bar{h})}{|\bar{h}|^0} = \lim_{\bar{h}\to\bar{0}} g(\bar{a}+\bar{h}) = g(\bar{a}) = 0 \ \checkmark.$$

(10)

In the inductive step, we use our old <u>telescoping</u> trick:

$$g(\bar a+\bar h) = g(\bar a+\bar h) - \underbrace{g(\bar a)}_{=0} = \sum_{i=1}^{n} g(\bar a+h_1\bar e_1+\dots+h_i\bar e_i) - g(\bar a+h_1\bar e_1+\dots+h_{i-1}\bar e_{i-1})$$



$$= \sum_{i=1}^{n} h_i \frac{\partial g}{\partial x_i}(\bar b_i)$$ for some choice of $\bar b_i \in [\bar a+h_1\bar e_1+\dots+h_{i-1}\bar e_{i-1},\ \bar a+h_1\bar e_1+\dots+h_i\bar e_i]$, $i=1,2,\dots,n$

Mean Value Thm in 1-variable

Hence to show $\lim_{\bar h\to\bar 0} \frac{g(\bar a+\bar h)}{|\bar h|^k}=0$, it suffices to show for each $i=1,2,\dots,n$ that

$$0 \overset{?}{=} \lim_{\bar h\to\bar 0} \frac{h_i \frac{\partial g}{\partial x_i}(\bar b_i)}{|\bar h|^k} = \lim_{\bar h\to\bar 0} \frac{h_i}{|\bar h|} \cdot \frac{\frac{\partial g}{\partial x_i}(\bar a+\bar c_i)}{|\bar h|^{k-1}}$$ where $\bar b_i = \bar a+\bar c_i$

$\frac{|h_i|}{|\bar h|} \leq 1$

approaches 0 as $\bar h\to\bar 0$ by induction on k, since $\bar h\to\bar 0$ forces $\bar c_i\to\bar 0$ ∎

This showed $f(\bar a+\bar h) - P^k_{f,\bar a}(\bar a+\bar h) \in o(|\bar h|^k)$.

If one had another polynomial P of deg $\leq k$ with this property, then $Q(\bar x) := P(\bar x) - P^k_{f,\bar a}(\bar x)$ would be ~~also~~ a polynomial of deg $\leq k$ having

$\lim_{\bar h\to\bar 0} \frac{Q(\bar a+\bar h)}{|\bar h|^k} = 0$ and it is easy to see this forces $Q(\bar x)=0$. ∎
(EXERCISE!)