(74) PROP 4.11.5 / THM 4.11.7 / DEF'N 4.11.8 : Suppose $\mathbb{R}^n \xrightarrow{f} \mathbb{R}$ has $f(\bar{x}) \underset{a.e}{\overset{(*)}{=}} \sum_{k=1}^{\infty} f_k(\bar{x})$

and that $\sum_{k=1}^{\infty} \int_{\mathbb{R}^n} |f_k(\bar{x})| \, |d^n\bar{x}|$ $\overset{(**)}{\text{converges}}$ (in particular, these $|f_k|$ are R-integrable $\forall k$)

Then we say f is L-integrable and define its

L-integral as $\int_{\mathbb{R}^n} f(\bar{x}) \, |d^n\bar{x}| := \underbrace{\sum_{k=1}^{\infty} \int_{\mathbb{R}^n} \underbrace{f_k(\bar{x})}_{\text{also R-integrable } \forall k} |d^n\bar{x}|}_{\text{converges absolutely, since } |\int f_k| \le \int |f_k|}$.

In particular, this will be independent of the choice of sequences $\{f_k\}$ satisfying $(*)$, $(**)$.

3/21/2017

EXAMPLES:

① R-integrable $\overset{\text{PROP 4.11.9}}{\Longrightarrow}$ L-integrable, since if f is R-integrable then one can always take $f_1 = f$, $f_2 = f_3 = f_4 = \ldots \equiv 0$ to make $(*)$ hold,

and $\sum_{k=1}^{\infty} \int |f_k(\bar{x})| \, |d^n\bar{x}| \overset{(**)}{=} \int \underbrace{|f_1(\bar{x})|}_{|f(x)| \text{ is R-integrable because } f(x) \text{ was}} |d^n\bar{x}|$ by PROP 4.1.14 (iv)

$\sum_{k=1}^{\infty} \int f_k(x) \, |d^n\bar{x}| = \int \underbrace{f_1(x)}_{f(x)} |d^n\bar{x}|$ ✓

② NON-EXAMPLE ③ from before, where $f = f_\infty = 1_{\mathbb{Q} \cap [0,1]}$ is now no problem, since we can list $\mathbb{Q} \cap [0,1] = \{a_1, a_2, a_3, a_4, a_5, \ldots\}$ as before
$$a_1 = 0,\ a_2 = 1,\ a_3 = \tfrac{1}{2},\ a_4 = \tfrac{1}{3},\ a_5 = \tfrac{2}{3}$$

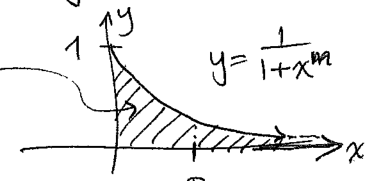
and write $f = \sum_{k=1}^{\infty} f_k$ where $f_k = 1_{\{a_k\}} = |f_k|$ has $\int_{\mathbb{R}} f_k(x) |dx| = \int_{\mathbb{R}} |f_k(x)| \, |dx| = 0$ (In fact, if we take $f_k \equiv 0 \, \forall k$ still have $f \underset{a.e.}{=} \sum_{k=1}^{\infty} f_k$!)

$\Rightarrow f = 1_{\mathbb{Q} \cap [0,1]}$ is L-integrable, with $\int_{\mathbb{R}} f(\bar{x}) \, |dx| = 0$

Similarly $f = 1_{\mathbb{Q}}$ or $f = 1_A$ for any subset $A \subset \mathbb{R}^n$ with measure 0 will be L-integrable, with $\int_{\mathbb{R}^n} f(\bar{x}) \, |d^n\bar{x}| = 0$.

③ Recall from 1-variable an improper integral like

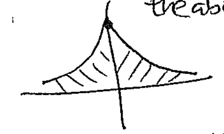
$$\int_0^\infty \frac{dx}{1+x^m} = \text{area here}$$



$$= \lim_{R\to\infty} \int_0^R \frac{dx}{1+x^m}$$

converges only when $m>1$. Let's show more generally

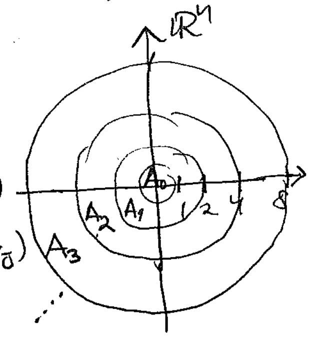
at $n=1$, it's twice the above:



that $f(\bar{x}) = \frac{1}{1+|\bar{x}|^m}$ on $\mathbb{R}^n$ is L-integrable

whenever $m>n$, by writing

$$f(\bar{x}) = \sum_{k=0}^{\infty} \underbrace{1_{A_k}(\bar{x}) \cdot \frac{1}{1+|\bar{x}|^m}}_{f_k(\bar{x})} \quad \text{where} \quad \begin{aligned} A_0 &= B_1(\bar{0}) \subset \mathbb{R}^n \\ A_1 &= B_{2^1}(\bar{0}) - B_1(\bar{0}) \\ A_2 &= B_{2^2}(\bar{0}) - B_{2^1}(\bar{0}) \\ &\vdots \\ A_k &= B_{2^k}(\bar{0}) - B_{2^{k-1}}(\bar{0}) \\ &\vdots \end{aligned}$$



and noting that for each $k \geq 1$ one has

$$\int_{\mathbb{R}^n} |f_k(\bar{x})| \, |d^n\bar{x}| = \int_{A_k} \frac{1}{1+|\bar{x}|^m} \, |d^n\bar{x}|$$

(linear) change-of-variable:

$$\mathbb{R}^n \xrightarrow{\Phi} \mathbb{R}^n$$

$$\bar{x} \longmapsto 2^{k-1}\bar{x}$$

$$\underbrace{\begin{bmatrix} 2^{k-1} & & 0 \\ & \ddots & \\ 0 & & 2^{k-1} \end{bmatrix}}_{n} \Big\}\, n$$

$$A_1 \xrightarrow{\Phi} A_k$$

$$= \int_{A_1} \frac{1}{1+|2^{k-1}\bar{x}|^m} \, |\det D\Phi(\bar{x})| \, |d^n\bar{x}|$$

$$= \int_{A_1} \frac{(2^{k-1})^n}{1+2^{m(k-1)}|\bar{x}|^m} \, |d^n\bar{x}|$$

$$\leq \int_{A_1} \frac{2^{n(k-1)}}{2^{m(k-1)}|\bar{x}|^m} \, |d^n\bar{x}| = 2^{(n-m)(k-1)} \underbrace{\int_{A_1} \underbrace{\frac{1}{|\bar{x}|^m}}_{\text{continuous on } A_1\text{, bounded, so R-integrable}} |d^n\bar{x}|}_{\text{call this integral } M}$$

and hence

$$\sum_{k=0}^{\infty} \int_{\mathbb{R}^n} |f_k(\bar{x})| \, |d^n\bar{x}| \leq M \underbrace{\sum_{k=1}^{\infty} 2^{(n-m)(k-1)}}_{\text{geometric series } \frac{1}{1-2^{n-m}}} + \underbrace{\int_{\mathbb{R}^n} |f_0(\bar{x})| \, |d^n\bar{x}|}_{\text{finite!}}$$

since $m>n$ i.e. $2^{n-m}<1$

(70)

L-integrals have some good, expected properties like...

easy; see pp. 506–507

PROP 4.11.14: (Linearity) If both f, g are L-integrable, so is $af+bg$ for $a,b\in\mathbb{R}$ and $\int_{\mathbb{R}^n}(af+bg)|d^n\bar{x}|$

PROP 4.11.15: $\left.\begin{array}{l} f \text{ L-integrable} \\ g \text{ R-integrable}\end{array}\right\} \Rightarrow fg$ L-integrable

proof: Write $f \underset{a.e.}{=} \sum_k f_k$ with $\sum_k \int |f_k|\,|d^n\bar{x}|$ convergent

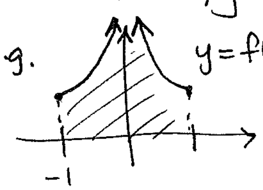
and let $M := \sup\{|g(\bar{x})| : \bar{x}\in\mathbb{R}^n\} < \infty$ since g must be bounded

Then $fg \underset{a.e.}{=} \sum_k \underbrace{f_k g}_{\text{each R-integrable}}$, with $\sum_k \int |f_k g|\,|d^n\bar{x}|$

$$\le \underbrace{\sum_k M \int |f_k|\,|d^n\bar{x}|}_{\text{convergent!}} \quad \blacksquare$$

REMARK: It can fail for f,g L-integrable, not R-integrable

e.g.



$y=f(x)=g(x)=\frac{1}{\sqrt{|x|}}\cdot 1_{[-1,1]}(x)$ is L-integrable,

but $f(x)g(x)=\frac{1}{|x|}\cdot 1_{[-1,1]}$ is not

3/22/2017 >

not hard; see pp. 507–508

PROP 4.11.16: f,g L-integrable with $f \underset{a.e.}{\le} g \Rightarrow \int_{\mathbb{R}^n} f\,|d^n\bar{x}| \le \int_{\mathbb{R}^n} g\,|d^n\bar{x}|$.

takes work; see App. A.22

THM 4.11.20 (Fubini) If $\overset{(\bar{x},\,\bar{y})}{\mathbb{R}^n\times\mathbb{R}^m} \xrightarrow{f} \mathbb{R}$ is L-integrable then $f_{\bar{x}}(\bar{y}) = \int_{\mathbb{R}^n} f(\bar{x},\bar{y})\,|d^n\bar{x}|$ is defined for almost $\bar{y}\in\mathbb{R}^m$, and L-integrable, and

$$\int_{\mathbb{R}^n\times\mathbb{R}^m} f(\bar{x},\bar{y})\,|d^n\bar{x}||d^m\bar{y}| = \int_{\mathbb{R}^m}\Big(\underbrace{\int_{\mathbb{R}^n} f(\bar{x},\bar{y})\,|d^n\bar{x}|}_{f_{\bar{x}}(\bar{y})}\Big)\,|d^m\bar{y}|$$

THM 4.11.21 (Change-of-variable) With suitable hypotheses on a parametrization $X \xrightarrow{\Phi} Y$ one has for $Y \xrightarrow{f} \mathbb{R}$ L-integrable that $X \xrightarrow{f\circ\Phi} \mathbb{R}$ is L-integrable, and

$$\int_X (f\circ\Phi)(\bar{x})\,|\det D\Phi(\bar{x})|\,|d^n\bar{x}| = \int_Y f(\bar{y})\,|d^n\bar{y}|$$