3/29/2017

(85)

Two issues:

① Is $\int_M f(\bar{x})\,|d^k\bar{x}|$ independendent of (nice) parametrization $\underset{\mathbb{R}^k}{\overset{\cap}{U}} \to \underset{\mathbb{R}^n}{\overset{\cap}{M}}$?

② What's a "nice" parametrization?

---

for ①, given 2 parametrizations

$$\begin{array}{ccccc} \mathbb{R}^k \supset U & \xrightarrow{\gamma_1} & M \subset \mathbb{R}^n & \xrightarrow{f} & \mathbb{R}, \\ \Phi := \gamma_2^{-1}\circ\gamma_1 \downarrow & \nearrow_{\gamma_2} \swarrow^{\gamma_2^{-1}} & & & \\ \mathbb{R}^k \supset V & & & & \end{array}$$

(PROP 5.3.3)

we want to check $\int_U (f\circ\gamma_1)(\bar{u})\sqrt{\det D\gamma_1(\bar{u})^T D\gamma_1(\bar{u})}\,|d^k\bar{u}| \overset{?}{=} \int_V (f\circ\gamma_2)(\bar{v})\sqrt{\det D\gamma_2(\bar{v})^T D\gamma_2(\bar{v})}\,|d^k\bar{v}|$

proof:

use $\bar{v} = \Phi(\bar{u})$ as a change-of-variable on $\mathbb{R}^k$!

$\underset{\mathbb{R}^k}{\overset{\cap}{U}} \xrightarrow{\Phi} \underset{\mathbb{R}^k}{\overset{\cap}{V}}$

$$\int_U \underbrace{(f\circ\gamma_2\circ\Phi)(\bar{u})} \sqrt{\det \cancel{D}D\gamma_2(\Phi(\bar{u}))^T D\gamma_2(\Phi(\bar{u}))} \cdot \underbrace{|\det D\Phi(\bar{u})|}_{\sqrt{\det D\Phi(\bar{u})^T D\Phi(\bar{u})}} \cdot |d^k\bar{u}|$$

$$\| \qquad\qquad\qquad\qquad \underbrace{\qquad\qquad\qquad\qquad\qquad\qquad}$$

$$(f\circ\gamma_2\circ\gamma_2^{-1}\circ\gamma_1)(\bar{u}) \qquad \sqrt{\det D\Phi(\bar{u})^T \cdot \det D\gamma_2(\Phi(\bar{u}))^T D\gamma_2(\Phi(\bar{u})) \cdot \det D\Phi(\bar{u})}$$

$$\| \qquad\qquad = \sqrt{\det\ D\Phi(\bar{u})^T D\gamma_2(\Phi(\bar{u}))^T\, D\gamma_2(\Phi(\bar{u}))D\Phi(\bar{u})}$$

$(f\circ\gamma_1)(\bar{u})$

as desired

$\boxed{\text{chain rule} + (AB)^T = B^T A^T}$

$$= \sqrt{\det\ D(\gamma_2\circ\Phi)(\bar{u})^T\ D(\gamma_2\circ\Phi)(\bar{u})}$$

$$= \sqrt{\det\ D\gamma_1(\bar{u})^T D\gamma_1(\bar{u})}$$

∎

as desired.

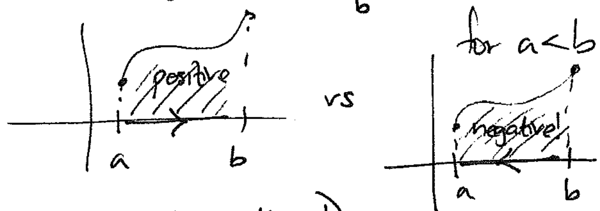
For ②, DEF'N 5.2.3 of relaxed parametrization of M by U is given in §5.2: $\underset{\mathbb{R}^k}{\overset{\cap}{U}} \xrightarrow{\gamma} \underset{\mathbb{R}^n}{\overset{\cap}{M}}$, $X \subset U$

with
- $U-X$ open
- γ continuous
- $\gamma(U) \supset M$
- $\gamma(U-X) \subset M$
- $U-X \xrightarrow{\gamma} M$ is injective, C^1, with locally Lipschitz $D\gamma$
- $D\gamma(\bar{u})$ injective $\forall \bar{u} \in U-X$
- $\mathrm{vol}_k(X) = 0$ and $\mathrm{vol}_k\, \gamma(X)\cap C = 0$ $\forall$ compact $C \subset M$

KEY POINTS:
- such parametrizations γ exist $\forall$ manifolds M (EXER. A.23.2 in Appendix) THM 5.2.8
- they make $\Phi: U \to V$, $\Phi = \gamma_2^{-1}\gamma_1$, a valid change-of-variables!

Chapter 6 Differential forms & vector calculus

Just as in 1-variable calculus, we distinguished $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$ for $a<b$



(whereas $\int_{\mathbb{R}^1} 1_{[a,b]}(x)\, f(x)\, |dx|$ didn't allow this!)

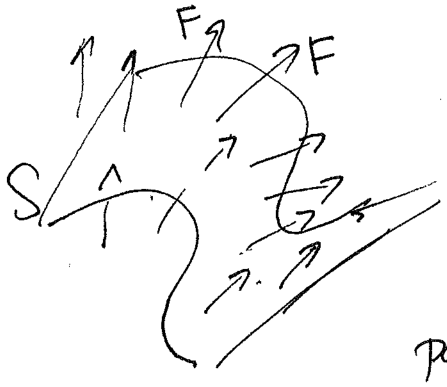
In many variables ...

- when we want to total ~~up~~ up the work done by a force field (wind, electric, etc.)

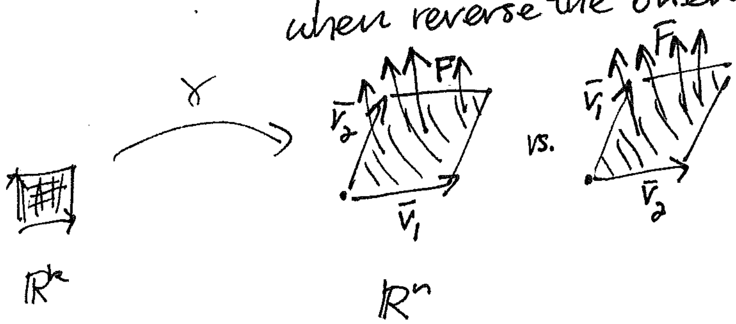
as we move along a curve C

we should pay attention to the direction in which we ~~trace~~/parametrize the curve (else the work done has opposite sign)

- when we total flux through a surface S (fluid flow, electric flux)



we should pay attention to the orientation of small parallelograms on the surface in our parametrization, and negate the flux when reverse the orientation



The key to this is differential forms. They take some getting used to, and it helps to compare them with div, grad, curl from vector calculus, which they capture as special cases.

Time to read Schey's book on this!

§6.1 Forms on $\mathbb{R}^n$

We want the quantities we're integrating to scale up as the parallelogram vectors $\bar{v}_1, \dots, \bar{v}_k$ scale up, and switch sign when we swap $\bar{v}_i$ & $\bar{v}_j$

DEF'N (6.1.1) A k-form on $\mathbb{R}^n$ is a function

$$(\mathbb{R}^n)^k = \mathbb{R}^n \times \mathbb{R}^n \times \dots \times \mathbb{R}^n \xrightarrow{\varphi} \mathbb{R}$$
$$(\bar{v}_1, \bar{v}_2, \dots, \bar{v}_k) \longmapsto \varphi(\bar{v}_1, \dots, \bar{v}_k)$$

which is multilinear in each position $i = 1, 2, \dots, k$

$$\varphi(\bar{v}_1, \dots, a\bar{v}_i + b\bar{v}_i', \dots, \bar{v}_k) = a\varphi(\bar{v}_1, \dots, \bar{v}_i, \dots, \bar{v}_k) + b\varphi(\bar{v}_1, \dots, \bar{v}_i', \dots, \bar{v}_k)$$

if $a, b \in \mathbb{R}$

and alternating

$$\varphi(\bar{v}_1, \dots, \bar{v}_i, \dots, \bar{v}_j, \dots, \bar{v}_k) = -\varphi(\bar{v}_1, \dots, \bar{v}_j, \dots, \bar{v}_i, \dots, \bar{v}_k)$$

for $1 \le i < j \le k$

EXAMPLES:

Ignore this one for the moment! [⓪ A 0-form is just a ~~function~~ real number $c \in \mathbb{R}$ $\underbrace{(\mathbb{R}^n)^0}_{\substack{\| \\ \{\emptyset\}}} \xrightarrow{\varphi} \mathbb{R}$ (?)]

$( ) \leftarrow$ the empty list of vectors!

$c = \varphi(\emptyset)$

① We've already seen an n-form on $\mathbb{R}^n$: the determinant

$$\mathbb{R}^n \times \mathbb{R}^n \times \dots \times \mathbb{R}^n \xrightarrow{\varphi = \det} \mathbb{R}$$
$$(\bar{v}_1, \bar{v}_2, \dots, \bar{v}_n) \longmapsto \varphi(\bar{v}_1, \dots, \bar{v}_n) = \det \begin{bmatrix} | & | & & | \\ \bar{v}_1 & \bar{v}_2 & \cdots & \bar{v}_n \\ | & | & & | \end{bmatrix}$$

= (signed) volume of $P(\bar{v}_1, \dots, \bar{v}_n)$

Recitation 3/30/2017?

② In fact, determinants also let us create some (and eventually, all) k-forms on $\mathbb{R}^n$ with $0 \le k \le n$ called the elementary k-forms $dx_{i_1} \wedge dx_{i_2} \wedge \dots \wedge dx_{i_k}$ for some ordered list $(i_1, i_2, \dots, i_k)$ with $i_j \in \{1, 2, \dots, n\}$

$$\mathbb{R}^n \times \dots \times \mathbb{R}^n \xrightarrow{dx_{i_1} \wedge \dots \wedge dx_{i_k}} \mathbb{R}$$
$$(\bar{v}_1, \dots, \bar{v}_k) \longmapsto \text{determinant of} \begin{bmatrix} - \text{row } i_1 - \\ - \text{row } i_2 - \\ \vdots \\ - \text{row } i_k - \end{bmatrix} \text{ of } n\left\{ \overbrace{\begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_k \\ | & | & & | \end{bmatrix}}^{k} \right.$$

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e.g. if $\bar{v}_1 = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 7 \end{bmatrix}$, $\bar{v}_2 = \begin{bmatrix} 1 \\ 1 \\ -2 \\ 3 \end{bmatrix}$ then $\begin{bmatrix} | & | \\ \bar{v}_1 & \bar{v}_2 \\ | & | \end{bmatrix} = \begin{array}{c} \begin{bmatrix} 2 & 1 \\ 3 & 1 \\ 4 & -2 \\ 7 & 3 \end{bmatrix} \end{array} \begin{array}{l} \leftarrow \boxed{\text{row } 1} \\ \\ \leftarrow \boxed{\text{row } 3} \\ \\ \end{array}$

and $dx_1 \wedge dx_3 (\bar{v}_1, \bar{v}_2) = \det \begin{bmatrix} 2 & 1 \\ 4 & -2 \end{bmatrix} = -8$

$dx_3 \wedge dx_1 (\bar{v}_1, \bar{v}_2) = \det \begin{bmatrix} 4 & -2 \\ 2 & 1 \end{bmatrix} = +8$

$dx_1 \wedge dx_3 (\bar{v}_1, \bar{v}_1) = \det \begin{bmatrix} 2 & 2 \\ 4 & 4 \end{bmatrix} = 0$

$dx_1 \wedge dx_4 (\bar{v}_1, \bar{v}_2) = \det \begin{bmatrix} 2 & 1 \\ 7 & 3 \end{bmatrix} = -1$

$dx_1 \wedge dx_4 (\bar{v}_2, \bar{v}_1) = \det \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix} = +1$

$dx_2 \wedge dx_2 (\bar{v}_1, \bar{v}_2) = \det \begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix} = 0$

Just as $dx_1 \wedge \dots \wedge dx_n (\bar{v}_1, \dots, \bar{v}_n) = \det \begin{bmatrix} | & & | \\ \bar{v}_1 & \cdots & \bar{v}_n \\ | & & | \end{bmatrix} = (\text{signed}) \, \mathrm{vol}_n (P(\bar{v}_1, \dots, \bar{v}_n))$,

one can also interpret $dx_{i_1} \wedge \dots \wedge dx_{i_k} (\bar{v}_1, \dots, \bar{v}_k)$

$= (\text{signed}) \, \mathrm{vol}_k P(\hat{v}_1, \hat{v}_2, \dots, \hat{v}_k)$

where $\hat{v}_1, \hat{v}_2, \dots, \hat{v}_k$ are the projections of $\bar{v}_1, \dots, \bar{v}_k$ onto the $x_{i_1}, x_{i_2}, \dots, x_{i_k}$ k-subspace in $\mathbb{R}^n$, (that is, $\mathrm{span}(\bar{e}_{i_1}, \bar{e}_{i_2}, \dots, \bar{e}_{i_k})$)

e.g. $(dx_1 \wedge dx_2)\left(\underset{\bar{v}_1}{\begin{bmatrix} a \\ b \\ c \end{bmatrix}}, \underset{\bar{v}_2}{\begin{bmatrix} d \\ e \\ f \end{bmatrix}} \right)$ = signed volume here
$= \det \begin{bmatrix} a & c \\ b & d \end{bmatrix} = ad - bc$