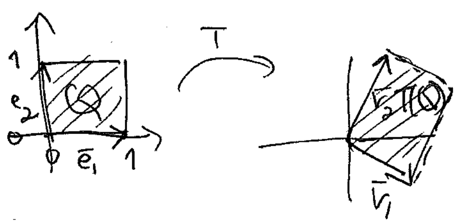
(66) 3/8/2017

4th step: Want to show $\mathrm{vol}_n \underbrace{T(Q)}_{\left\{\sum_{i=1}^n c_i\bar{v}_i \,:\, c_i\in[0,1]\right\}} = |\det A|$ if $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^n$, $\bar{x} \longmapsto A\bar{x}$

where $A = \begin{bmatrix} | & & | \\ \bar{v}_1 & \cdots & \bar{v}_n \\ | & & | \end{bmatrix}$



Key idea: It suffices for us to factor $T = E_1E_2\cdots E_k$ into elementary matrices, and show it's true only for elementary matrices $T = E_k$, since then we'd have via induction on k,

that $\mathrm{vol}_n T(Q) = \mathrm{vol}_n E_1E_2\cdots E_{k-1}(E_k(Q))$

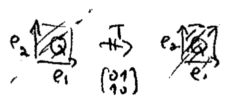
$\boxed{\text{use 3rd step with } T = E_1E_2\cdots E_{k-1}}$

$= \mathrm{vol}_n E_1E_2\cdots E_{k-1}(Q)\cdot \mathrm{vol}_n E_k(Q)$

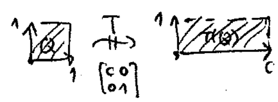
$= \underbrace{|\det(E_1E_2\cdots E_{k-1})|}_{\text{via induction}} \cdot \underbrace{|\det(E_k)|}_{\text{via the case } T=E_k \text{ elementary}} = |\det T|$

For elementary matrices E

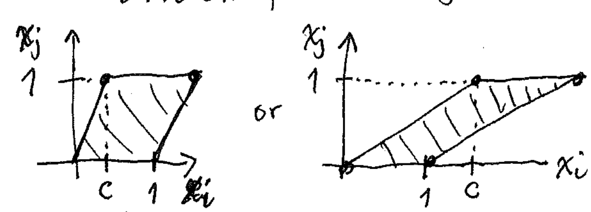
— of the form $E = \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & 0 & 1 & \\ & & 1 & 0 & \\ & & & & \ddots \\ & & & & & 1 \end{bmatrix}$ (col i, col j), it's easy since $|\det E| = |-1| = +1$ and $E(Q) = Q$ since E just permutes $\bar{e}_i$ and $\bar{e}_j$.

e.g. 

— of the form $E = \begin{array}{c} \\ \text{row } i \rightarrow \\ \\ \end{array}\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & c & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}$ (col i), it's easy since $|\det E| = c$ and $E(Q)$ is a rectangular box $[0,1]\times\cdots\times[0,1]\times[0,c]\times[0,1]\times\cdots\times[0,1]$ whose volume we computed as $1\cdots 1\cdot c\cdot 1\cdots 1 = c$.

e.g. 

— of the form $E = \begin{array}{c} \text{row } i \rightarrow \\ \\ \\ \end{array}\begin{array}{c} \text{col } j \\ \downarrow \\ \begin{bmatrix} 1 & & c & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \end{array}$, it takes a tiny geometric argument: $E(Q)$ is the product of $\overbrace{[0,1]\times\cdots\times[0,1]}^{\text{coordinates } \neq i,j}$ with this parallelogram:



both have area (base)(height) $= 1\cdot 1 = 1$ and $|\det E| = 1$. ∎

§4.10 Change-of-variables formula

Very important! Makes hard integrals (sometimes) calculable.

A warm-up:

THM 4.9.7. (Linear change-of-variable) If $\underset{\bar{y}}{\mathbb{R}^n} \xrightarrow{f} \underset{f(\bar{y})}{\mathbb{R}}$ is integrable and $\underset{\bar{x}}{\mathbb{R}^n} \xrightarrow{T} \underset{T\bar{x}=\bar{y}}{\mathbb{R}^n}$ linear and invertible, then $f\circ T$ is integrable, and

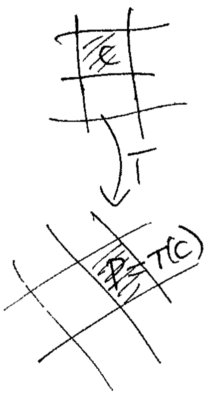
$$\int_{\mathbb{R}^n} f(\bar{y})\,|d^n\bar{y}| = |\det T| \int_{\mathbb{R}^n} \underbrace{f(T(\bar{x}))}_{(f\circ T)(\bar{x})}\,|d^n\bar{x}|$$

(think like a linear substitution $y = T(x) = cx$ in 1-variable $\int f(y)\,dy = \int f(T(x))\,\underbrace{T'(x)}_{c}\,dx$)

$dy = \underbrace{T'(x)}_{}dx = c\,dx$

$T'(x) = T(x)$ since T is linear

$= \det T$ since $\mathbb{R}^1 \xrightarrow{T} \mathbb{R}^1$



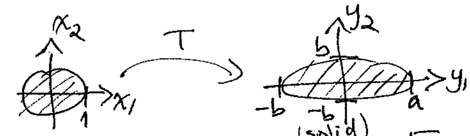
proof: $|\det T| \int_{\mathbb{R}^n} f(T(\bar{x}))\,|d^n\bar{x}| = \lim_{N\to\infty} \sum_{C\in\mathcal{D}_N(\mathbb{R}^n)} M_C(f\circ T)\,\underbrace{|\det T|\,\mathrm{vol}_n(C)}_{\mathrm{vol}_n T(C)}$

$$= \lim_{N\to\infty} \sum_{\substack{P\in T(\mathcal{D}_N(\mathbb{R}^n))\\ \overset{\shortparallel}{T(C)}}} M_P(f)\,\mathrm{vol}_n(P) = \int_{\mathbb{R}^n} f(\bar{y})\,|d^n\bar{y}|$$

Same calculation works with $m_C(f\circ T)$, $m_P(f)$, so $f\circ T$ is integrable, and $\int_{\mathbb{R}^n} (f\circ T)(\bar{x})\,|d^n\bar{x}| = \frac{1}{|\det T|}\int_{\mathbb{R}^n} f(\bar{y})\,|d^n\bar{y}|$ ∎

EXAMPLE: $\mathbb{R}^2 \xrightarrow{T} \mathbb{R}^2$ takes unit circle (disk) C

$\bar{x} \longmapsto \underbrace{\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}}_{A}\bar{x}$



(solid) ellipse E w/ axes a, b

$\frac{y_1^2}{a^2} + \frac{y_2^2}{b^2} \le 1$

and hence

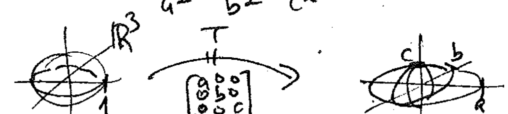
$$\mathrm{Area}(\underset{E}{\text{solid ellipse}}) = \int_{\mathbb{R}^2} 1_E(\bar{y})\,|d^2\bar{y}| = \left|\det\begin{bmatrix} a & 0 \\ 0 & b\end{bmatrix}\right| \int_{\mathbb{R}^2} (1_E\circ T)(\bar{x})\,|d^2\bar{x}|$$

$$= |ab| \int_{\mathbb{R}^2} 1_C(\bar{x})\,|d^2\bar{x}| = ab\,(\text{area of circle})$$

$$= \pi ab.$$

Similarly, the (solid) ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1$ has volume $\frac{4}{3}\pi abc$

since



unit ball of volume $\frac{4}{3}\pi$ — det=abc — solid ellipsoid

(68) In general, change-of-variables looks like this:

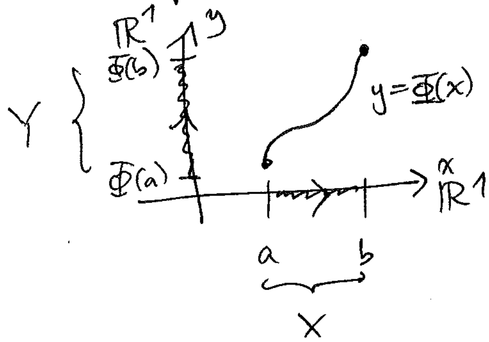
THM 4.10.2 (roughly)

Under hypotheses saying $Y \subset \mathbb{R}^n$ is, roughly speaking, nicely parametrized by $\begin{matrix} X & \xrightarrow{\Phi} & Y \\ \cap & & \cap \\ \mathbb{R}^n & & \mathbb{R}^n \end{matrix}$,

any integrable $Y \xrightarrow{f} \mathbb{R}$ has

$$\int_Y f(\bar{y})\,|d^n\bar{y}| = \int_X (f\circ\Phi)(\bar{x})\,\big|\det[D\Phi(x)]\big|\,|d^n\bar{x}|$$

Compare with 1-variable picture of substitution



$$\int_{y=\Phi(a)}^{y=\Phi(b)} f(y)\,dy = \int_{x=a}^{x=b} f(\Phi(x))\,\underbrace{\Phi'(x)}\,dx$$

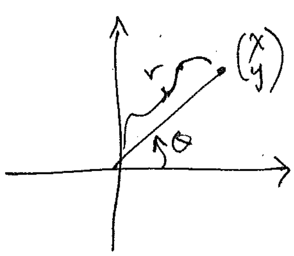
positive, if Φ is monotone increasing on $X=[a,b]$

$y=\Phi(x)$
$dy=\Phi'(x)dx$

3/10/2017 EXAMPLES:

① Polar coordinates (DEF'N 4.10.2, PROP 4.10.3) If $\mathbb{R}^2 \xrightarrow{\Phi} \mathbb{R}^2$

$$\begin{pmatrix} r \\ \theta \end{pmatrix} \longmapsto \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}$$



maps $\begin{matrix} B & \longrightarrow & A \\ \cap & & \cap \\ [0,\infty)\times[0,2\pi) & & \mathbb{R}^2 \end{matrix}$ nicely, then

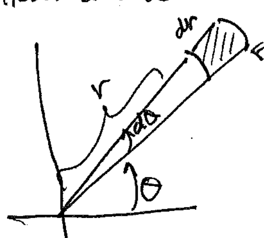
$$\int_A f\begin{pmatrix} x \\ y \end{pmatrix} |dx\,dy| = \int_B f\begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix} r\,|dr\,d\theta|$$

because $|\det[D\Phi]| = \left|\det\begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix}\right| = \left|\det\begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}\right|$

$= |r\cos^2\theta + r\sin^2\theta|$

$= |r| = r$

Heuristic derivation of $r\,dr\,d\theta$:



$\leftarrow$ area $= \frac{d\theta}{2\pi} \times (\pi(r+dr)^2 - \pi r^2)$

$\approx \cancel{\pi r^2} + 2\pi r\,dr + \pi(dr)^2 - \cancel{\pi r^2}$ (negligible)

$\approx \frac{d\theta}{2\pi}\cdot 2\pi r\,dr$

$= r\,dr\,d\theta$