Math 4707 Intro to combinatorics and graph theory
Fall 2016, Vic Reiner
Midterm exam 1- Due Wednesday Oct 19, in class

Instructions: There are 5 problems, worth 20 points each, totaling 100 points. This is an open book, open library, open notes, open web, take-home exam, but you are *not* allowed to collaborate. The instructor is the only human source you are allowed to consult.

1. (20 points total)

(a) (5 points) How many rearrangements are there of the letters in the word "COMMITTEE"

(b) (15 points) What is the probability that such a rearrangement has no identical letters consecutive (no "MM", "TT" nor "EE")?

2. (20 points) Recall that the Fibonacci numbers are defined by a recurrence $F_{n+1} = F_n + F_{n-1}$, with initial conditions $F_0 = 0, F_1 = 1$.

Without using the recurrence to compute $F_{1,000,000,000}$ explicitly, predict how many decimal digits it will contain, up to an error of 2 digits. Explain why your answer is correct to within 2 digits.
(Hint: recall that we know an exact formula for F_n).

3. (20 points total; 10 points each part) Recall that $a \equiv b \bmod m$, or a is congruent to b modulo m, means that a, b have the same remainder upon division by m, or that $a - b$ is a multiple of m.

(a) Fill in the blanks that make the following conjecture correct:

Conjecture: *The Fibonacci numbers (defined as in Problem 2) have*

$$F_n \equiv \begin{cases} 0 \bmod 3 & \textit{if } n \equiv \underline{?} \textit{ or } \underline{?} \bmod 8 \\ 1 \bmod 3 & \textit{if } n \equiv \underline{?} \textit{ or } \underline{?} \textit{ or } \underline{?} \bmod 8 \\ 2 \bmod 3 & \textit{if } n \equiv \underline{?} \textit{ or } \underline{?} \textit{ or } \underline{?} \bmod 8 \end{cases}$$

(b) Prove your conjecture.

4. (20 points) Exercise 1.8.29 on p. 24 of our text: In how many ways can one color n distinct objects (labeled $1, 2, \dots, n$) with 3 colors, if each color must be used at least once?
(Your answer should be expressed as a function of n.)

5. (20 points) Exercise 1.8.32 on p. 24 of our text: Find all triples (a, b, c) of positive integers with $a \geq b \geq c \geq 1$ such that

$$\binom{a}{b}\binom{b}{c} = 2\binom{a}{c}.$$