Math 4707 Intro to combinatorics and graph theory
Spring 2008, Vic Reiner
Final exam- Due Wednesday May 7, in class

Instructions: This is an open book, open library, open notes, open web, take-home exam, but you are *not* allowed to collaborate. The instructor is the only human source you are allowed to consult.

1. (20 points total) Consider all the rearrangements of the letters in the word COMMITTEE into (possibly nonsensical) words. For example, CEEIMMOTT is one such word.

(a) (5 points) How many such words are there?

(b) (5 points) How many are there which do not have the two M's adjacent?

(c) (10 points) How many are there which have no double letters adjacent, i.e. no two M's adjacent and no two T's adjacent and no two E's adjacent.

2. (20 points) Define a generating function t_n that keeps track of labelled trees T on vertex set $V = \{1, 2, \ldots, n\}$ according to the vertex-degrees $\deg_T(i)$ in the following way:

$$t_n(x_1, x_2, \ldots, x_n) := \sum_{\text{such trees } T} x_1^{\deg_T(1)} \cdots x_n^{\deg_T(n)}.$$

For example, when $n = 2$, there is only one such tree T, with a single edge, having $\deg_T(1) = \deg_T(2) = 1$. Hence

$$t_2(x_1, x_2) = x_1 x_2.$$

When $n = 3$, there are three such trees, call them T_1, T_2, T_3, each isomorphic to a path with 2 edges, in which T_i has the vertex i "in the middle" of the path. One finds then that

$$t_3(x_1, x_2, x_3) = x_1^2 x_2 x_3 + x_1 x_2^2 x_3 + x_1 x_2 x_3^2 = x_1 x_2 x_3 (x_1 + x_2 + x_3).$$

(a) (5 points) What is the evaluation

$$t_n(1, 1, \ldots, 1) = [t_n(x_1, x_2, \ldots, x_n)]_{x_1 = x_2 = \cdots = x_n = 1}$$

as a function of n?

(b) (5 points) Every monomial $x_1^{d_1} \cdots x_n^{d_n}$ which appears in the polynomial $t_n(x_1, \ldots, x_n)$ has exactly the same degree. E.g. for $n = 2$, this degree was 2, and for $n = 3$ this degree was 4. What is this degree, as a function of n?

(c) (10 points) Prove that

$$t_n(x_1, x_2, \ldots, x_n) := x_1 x_2 \cdots x_n (x_1 + x_2 + \cdots + x_n)^{n-2}$$

for every $n \geq 2$.

3. (20 points) A usual soccer ball is made up of 12 pentagons and some number of hexagons, sewn together so that exactly 3 polygons meet at each vertex.

Your boss in the design department of the Beijing soccer ball factory wants you to design a *new* soccer ball for the olympics, again with exactly 3 polygons meeting at each vertex, but this time made up of only *squares* and hexagons. He insists that there should be at least 12 squares used in the ball.

Convince him (tactfully) that he is wrong, by deriving the exact number of squares that must be used in any such hypothetical soccer ball.

4. (20 points total) Given a graph $G = (V, E)$ with no loops and no multiple edges, informally, its *line graph* is the graph with one vertex v_e for each edge e of G, and two such vertices $v_e, ve_{e'}$ connected by an edge of $\text{Line}(G)$ if $\text{th}e, e'$ share any endpoints in G. More formally, $\text{Line}(G) = (V_{\text{Line}(G)}, E_{\text{Line}(G)})$ with

$$\begin{aligned} V_{\text{Line}(G)} &= \{v_e : e \in E\} \\ E_{\text{Line}(G)} &= \{\{v_e, v_{e'}\} : e, e' \text{ have a common endpoint}\}. \end{aligned}$$

For example, if G is a cycle of size n, then $\text{Line}(G)$ is also isomorphic to a cycle of size n.

For another example, if G is the complete graph K_4, then $\text{Line}(G)$ is isomorphic to vertices and edges of an octahedron (the 8-sided Platonic solid).

For a third example, see the one pictured on the Wikipedia page `en.wikipedia.org/wiki/Line_graph`.

(a) (10 points) Given an edge $e = \{v, v'\}$ in G what is the degree of the vertex v_e in Line(G), written as a function of $\deg_G(v)$ and $\deg_G(v')$?

(b) (10 points) Let G be a graph with no isolated vertices. Prove that if G has an Euler tour (that is, a closed Euler walk, that is, a walk that goes over every edge exactly once and returns to its starting vertex) then Line(G) also has an Euler tour.

5. (20 points total) Given a graph $G = (V, E)$ with no loops, recall from class that an *acyclic orientation* of G is an assignment of one of the two possible orientations to every edge e in E, so as to make G a directed graph, but so as to create no directed cycles in this digraph.

Recall also that the *chromatic polynomial* $\chi(G, k)$ was defined to be the polynomial in k that gives as its value for positive integers k the number of proper vertex-colorings of G with k colors.

(a) (5 points) Prove that multiple edges in G do not affect the number of acyclic orientations of G, that is, if $\overline{G}$ is obtained from G by retaining only one copy of each edge that has the same endpoints $\{v, v'\}$, then $G, \overline{G}$ have the same number of acyclic orientations.

(b) (5 points) Show that the number of acyclic orientations for a tree T on n vertices does not depend upon the structure of the tree itself, by writing this number as a function of n.

(c) (10 points) Show that the chromatic polynomial $\chi(T, k)$ for a tree T on n vertices does not depend upon the structure of the tree itself, by writing down the polynomial $\chi(T, k)$ as a function of n.