Math 4707 Intro to combinatorics and graph theory
Spring 2008, Vic Reiner
Midterm exam 2- Due Wednesday April 16, in class

Instructions: This is an open book, open library, open notes, open web, take-home exam, but you are *not* allowed to collaborate. The instructor is the only human source you are allowed to consult.

1. (30 points total; 10 points each part) Recall that a *forest* is a graph $G = (V, E)$ containing no cycles, and a tree is a connected forest.

(a) What is the number of edges $|E|$ in a forest that has $|V| = n$ vertices and exactly c different connected components?

(b) Prove that in a tree on vertex set $V = \{1, 2, \ldots, n\}$, if d_i denotes the degree of vertex i, then

$$\sum_{i=1}^{n} (d_i - 1) = n - 2.$$

(c) Prove that given any set of nonnegative integers d_i that satisfy the equation in part (b), the number of different (labelled) trees on vertex set $V = \{1, 2, \ldots, n\}$ in which vertex i has degree d_i is the multinomial coefficient

$$\binom{n-2}{d_1 - 1,\ d_2 - 1,\ \ldots,\ d_n - 1}.$$

2. (20 points) Prove that the number of *unlabelled* trees on n vertices (that is, isomorphism classes of trees on n vertices) is at most $\binom{2n-2}{n-1}$. (Hint: how did we already get an upper bound, in lecture or in the book, on the number of such unlabelled trees?)

3. (30 points total) Recall that K_n denotes the *complete graph* on vertex set $\{v_1, v_2, \ldots, v_n\}$, having an edge $\{v_i, v_j\}$ for each pair $1 \leq i < j \leq n$. Recall also that $K_{m,n}$ denotes the *complete bipartite graph* on bipartite vertex set $X \sqcup Y = \{x_1, \ldots, x_m\} \sqcup \{y_1, \ldots, y_n\}$, having an edge $\{x_i, y_j\}$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. Answer each of the following, and prove your answer in each case.

(a) (5 points) For which values of $n \geq 2$ does K_n have an Euler tour (= a closed Eulerian walk in our book's terminology)?

(b) (5 points) For which values of $n \geq 2$ does K_n have a Hamilton cycle?

(c) (10 points) For which values of $m, n \geq 2$ does $K_{m,n}$ have an Euler tour?

(d) (10 points) For which values of $m, n \geq 2$ does $K_{m,n}$ have a Hamilton cycle?

4. (20 points) Your company has 6 employees $\{x_1, \ldots, x_6\}$ and 6 tasks to perform $\{y_1, \ldots, y_6\}$, but each employee has a different set of tasks they are capable of doing:

employee	tasks they can do
x_1	$\{y_2, y_4, y_5\}$
x_2	$\{y_1, y_2, y_3, y_5, y_6\}$
x_3	$\{y_2, y_4, y_5\}$
x_4	$\{y_2, y_4\}$
x_5	$\{y_2, y_4, y_5\}$
x_6	$\{y_1, y_3, y_5, y_6\}$

Match each employee to at most one task, so that different employees end up doing different tasks, in such a way that the maximum number of tasks are performed. Prove that your answer attains the maximum.