Math 4707 Intro to combinatorics and graph theory
Spring 2017, Vic Reiner
Midterm exam 2- Due Monday April 10, in class

Instructions: There are 5 problems, worth 20 points each, totaling 100 points. This is an open book, open library, open notes, open web, take-home exam, but you are *not* allowed to collaborate. The instructor is the only human source you are allowed to consult.

1. (20 points total) Recall that a *forest* is a graph containing no cycles, that a *tree* is a connected forest, and a *leaf* is a vertex of degree one.

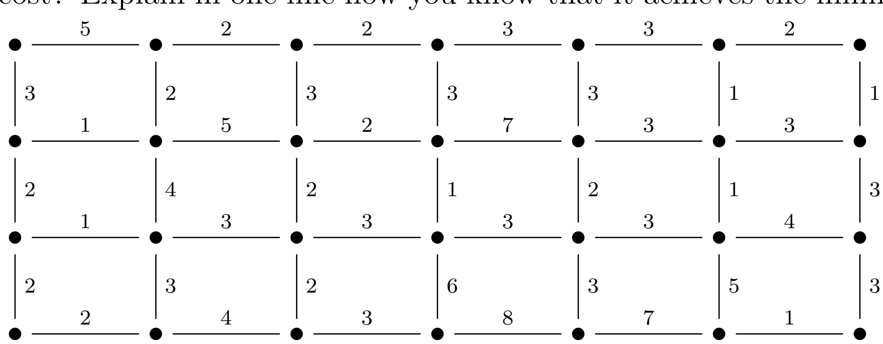
(a) (10 points) Prove that a tree with at least one degree d vertex has at least d distinct leaves.

(b) (5 points) Prove that a tree T with n vertices has

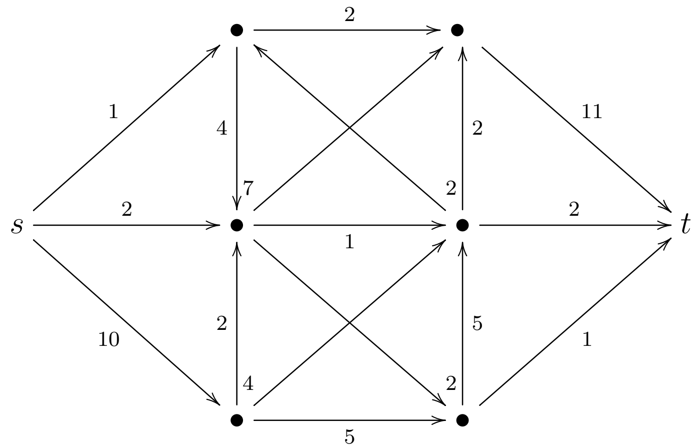
$$\sum_{v \in V} (\deg_T(v) - 1) = n - 2.$$

(c) (5 points) Given a forest with n vertices and c connected components, how many edges will it contain (as a function of n and c)?

2. (20 points total) Find (darken) a minimum cost spanning tree T in this graph whose edges have been labeled by their costs. What is its cost? Explain in one line how you know that it achieves the minimum.



3. (20 points) Find a maximum-valued flow from s to t in the flow network shown below, whose arcs have been labeled by their flow capacities. Explain how you know that it achieves the maximum value.



4. (20 points total; 10 points each part)
Let $G = (V, E)$ be bipartite graph, with vertex partition $V = X \sqcup Y$. Assume further that

- every x in X has the same degree $d_X \geq 1$, and
- every y in Y has the same degree $d_Y \geq 1$.

(a) Prove that $\frac{d_X}{d_Y} = \frac{|Y|}{|X|}$.

(b) Assuming without loss of generality that $d_X \geq d_Y$, show that there exists at least one matching $M \subset E$ with number of edges $|M| = |X|$.

5. (20 points) (Problem 7.2.11 on p. 134 of our text)
Prove that a graph G with *no multiple edges and no self-loops* having n vertices and strictly more than $\binom{n-1}{2}$ edges must be connected.