Math 5251 Error-correcting codes and finite fields, Fall 2024,
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Midterm exam 2
Due Wednesday November 13 by 11:59pm, via Canvas

Instructions: This is an open book, open notes, open web, take-home exam, but you may *not* collaborate. You must **clearly indicate** any such sources used, including AI sources such as ChatGPT. The instructor is the only human source that you are allowed to consult.

1. (25 points total) Let p be a prime number, and $\mathcal{C}$ an $\mathbb{F}_p$-linear code with a generator matrix

$$H = \begin{bmatrix} \overline{3} & \overline{4} & \overline{5} & \cdots & \overline{m-1} & \overline{m} \end{bmatrix}$$

for some integer m with $5 \leq m \leq p-1$.

(a) (15 points) Compute the three parameters $[n,k,d]$ for $\mathcal{C}$ as a linear code, and justify your answer.

(b) (10 points) Compute the three parameters $[n,k,d]$ for its dual code $\mathcal{C}^{\perp}$, and justify your answer.

2. (25 points) For each prime number p, either construct explicitly a linear code $\mathcal{C}$ having $|\mathcal{C} \cap \mathcal{C}^{\perp}| = p$, or prove that none can exist.

3. (25 points) Let a, b, c be distinct positive integers, and p a prime number. Does x^5 have a multiplicative inverse in $\mathbb{F}_p[x]/(1+x^a+x^b+x^c)$? Prove your answer.

4. (25 points total)
(a) (5 points) Compute $GCD(41, 100000)$, showing your work.

(b) (5 points) Compute $(\overline{41})^{-1}$ in $\mathbb{Z}/100000$, showing your work.

(c) (5 points) Compute $(\overline{100000})^{-1}$ in $\mathbb{Z}/41$, showing your work.

(d) (10 points) Let N be a positive integer written in decimal notation

$$N = \cdots d_{14}d_{13}d_{12}d_{11}d_{10}d_9d_8d_7d_6d_5d_4d_3d_2d_1d_0,$$

meaning that d_0 is the ones digits, d_1 is the tens digits, d_2 is the hundreds digit, etc. Prove that in $\mathbb{Z}/41$, one has

$$\overline{N} = \cdots + \overline{d_{14}d_{13}d_{12}d_{11}d_{10}} + \overline{d_9d_8d_7d_6d_5} + \overline{d_4d_3d_2d_1d_0},$$

where each of $d_4d_3d_2d_1d_0$, $d_9d_8d_7d_6d_5$, $d_{14}d_{13}d_{12}d_{11}d_{10}$, $\ldots$ is again a number written in decimal notation.