Math 5285 Honors abstract algebra
Fall 2007, Vic Reiner
Midterm exam 1- Due Wednesday December 12, in class

Instructions: This is an open book, open library, open notes, open web, take-home exam, but you are *not* allowed to collaborate. The instructor is the only human source you are allowed to consult.

1. (10 points) Artin's Exercise 2.4.19 on page 73.

2. (20 points total; 5 for each part)
Let T be the linear transformation $\mathbb{R}^2 \to \mathbb{R}^2$ which fixes the origin and rotates a vector through an angle of θ counterclockwise, represented by the following matrix with respect to the standard basis vectors (e_1, e_2) for $\mathbb{R}^2$:

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Recall that to *diagonalize* A over $\mathbb{R}$ means to find a matrix $P \in GL_2(\mathbb{R})$ for which $A' := PAP^{-1}$ is diagonal.

(a) Prove that one can diagonalize A over $\mathbb{R}$ if and only if θ is a multiple of π.

(b) Consider the same matrix A as an element of $\mathbb{C}^{2\times 2}$, that is, as having complex number entries. Prove that it can always be diagonalized *over* $\mathbb{C}$, regardless of the choice of θ.

(c) Let T be rotation in $\mathbb{R}^3$, with rotation axis passing through the origin in the direction of a nonzero vector $v \in \mathbb{R}^3$, and rotating through an angle of $\frac{\pi}{2}$ (i.e. 90 degrees) about this axis. Describe a basis for $\mathbb{R}^3$ and a matrix $A \in \mathbb{R}^{3\times 3}$ that represents T with respect to this basis.

(d) Consider the same matrix in part (c) as lying in $\mathbb{C}^{3\times 3}$, and diagonalize it over $\mathbb{C}$.

3. (15 points total; 5 for part (a), 10 for part (b))
Prove that inside $GL_n(\mathbb{F}_p)$ for p prime, the subset consisting of upper triangular matrices having all 1's on the diagonal is ...

(a) a subgroup, and

(b) a p-Sylow subgroup.

4. (20 points total; 5 for part (a), 5 for part (b), 10 for part (c))
Let p be a prime.

(a) Show that any element A in $GL_n(\mathbb{F}_p)$ has finite order.

(b) Show that a diagonal element

$$D = \begin{bmatrix} c_1 & & & \\ & c_2 & & \\ & & \ddots & \\ & & & c_n \end{bmatrix}$$

in $GL_n(\mathbb{F}_p)$ will have order equal to the least common multiple

$$LCM(d_1, d_2, \ldots, d_n)$$

in which d_i is the multiplicative order of c_i inside the group $\mathbb{F}_p^\times$.

(c) Prove that if A in $GL_n(\mathbb{F}_p)$ has order p then it is *not* diagonalizable.

5. (20 points total; 10 for each part)
Recall that for a ring R, we let R^+ denote the abelian group structure coming from the addition in R, ignoring multiplication.

(a) Prove that for every positive integer m, any group homomorphism $\phi : (\mathbb{Z}/m\mathbb{Z})^+ \to \mathbb{Z}^+$ must be the zero homomorphism, that is, $\phi(x) = 0$ for all x in $\mathbb{Z}/m\mathbb{Z}$.

(b) Prove that any group homomorphism $\phi : \mathbb{Q}^+ \to \mathbb{Z}^+$ must be the zero homomorphism that is, $\phi(x) = 0$ for all x in $\mathbb{Q}$.

6. (15 points total; 5 points each for parts (a),(b),(c); no credit for (d))
Recall these formulas for the binomial coefficient

$$\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} = \frac{n!}{k!(n-k)!}$$

counting the number of k-elements subset of an n-element set N.

(a) Prove the first formula by considering the two sets

$T = \{k\text{-element } \textit{ordered} \text{ sequences } (a_1, \ldots, a_k) \text{ of distinct elements in } N\}$
$S = \{k\text{-element (unordered) subsets } \{a_1, \ldots, a_k\} \subset N\}$

and showing that the map

$$\begin{array}{ccc} T & \xrightarrow{f} & S \\ (a_1, \ldots, a_k) & \longmapsto & \{a_1, \ldots, a_k\} \end{array}$$

has every fiber $f^{-1}(\{a_1, \ldots, a_k\})$ of the same cardinality. That is, explain how knowing this fiber cardinality and the cardinality of T leads to the formula for the cardinality of S.

(b) Prove the second formula by considering the following action of the symmetric group $G = S_n$ on S

$$\begin{array}{rcl} S_n \times S & \to & S \\ (p, \{a_1, \ldots, a_k\}) & \mapsto & p\{a_1, \ldots, a_k\} := \{p(a_1), \ldots, p(a_k)\} \end{array}$$

and applying the counting formula $|G| = |O_s||G_s|$.

Now let p be a prime, and define the *p-binomial coefficient*

$$\begin{bmatrix} n \\ k \end{bmatrix}_p := \frac{(p^n-1)(p^n-p)(p^n-p^2)\cdots(p^n-p^{k-1})}{(p^k-1)(p^k-p)(p^k-p^2)\cdots(p^k-p^{k-1})} = \frac{n!_p}{p^{k(n-k)} \cdot k!_p \cdot (n-k)!_p}$$

in which we are using the notation

$$n!_p := (p^n-1)(p^n-p)(p^n-p^2)\cdots(p^n-p^{n-1}).$$

Let V be an n-dimensional vector space over $\mathbb{F}_p$, and define

$T = \{k\text{-element } \textit{ordered} \text{ linearly indepent sets } (v_1, \ldots, v_k) \text{ inside } V\}$

$S = \{k\text{-dimensional subspaces } W \subset V\}$.

(c) Show that S has cardinality $|S| = \begin{bmatrix} n \\ k \end{bmatrix}_p$ by showing that the map

$$\begin{array}{rcl} T & \xrightarrow{f} & S \\ (v_1, \ldots, v_k) & \longmapsto & W := \mathrm{Span}\{v_1, \ldots, v_k\} \end{array}$$

has every fiber $f^{-1}(W)$ of the same cardinality. That is, explain how knowing this fiber cardinality and the cardinality of T leads to the first formula for $\begin{bmatrix} n \\ k \end{bmatrix}_p$ as the cardinality of S.

(d) (**Just for fun if you feel like it; not for credit**) Can you show $|S|$ is given by the second formula for $\begin{bmatrix} n \\ k \end{bmatrix}_p$, by considering the following action of the general linear group $G = GL_n(\mathbb{F}_p)$ on S

$$\begin{array}{rcl} GL_n(\mathbb{F}_p) \times S & \to & S \\ (g, W) & \mapsto & g(W) \end{array}$$

and applying the counting formula $|G| = |O_s||G_s|$?