Math 5285 Honors fundamental structures of algebra
Fall 2018, Vic Reiner
Final exam - Due Wednesday December 12, in class

Instructions: There are 4 problems. This is an open book, open library, open notes, open web, take-home exam, but you are *not* allowed to collaborate. The instructor is the only human source you are allowed to consult.

1. (55 points total, 5 points each) Prove, or disprove the following assertions.

(a) For any prime $p \geq 3$, let $H := \{A \in (\mathbb{F}_p)^{n \times n} : \det(A)^2 = 1 \text{ in } \mathbb{F}_p\}$. Then H is a subset of $GL_n(\mathbb{F}_p)$.

(b) The set H described in part (a) is a subgroup of $GL_n(\mathbb{F}_p)$.

(c) The set H described in part (a) is a normal subgroup of $GL_n(\mathbb{F}_p)$.

(d) For all positive integers n, the only group homomorphism $(\mathbb{Z}/n\mathbb{Z})^+ \longrightarrow \mathbb{Z}^+$ is the homomorphism sending all elements to 0.

(e) The only group homomorphism $\mathbb{Q}^+ \longrightarrow \mathbb{Z}^+$ is the homomorphism sending all elements to 0.

(f) For all positive integers n, the only group homomorphism $(\mathbb{Z}/n\mathbb{Z})^\times \longrightarrow \mathbb{Z}^\times$ is the homomorphism sending all elements to $+1$.

(g) The only group homomorphism $\mathbb{Q}^\times \longrightarrow \mathbb{Z}^\times$ is the homomorphism sending all elements to $+1$.

(h) There do not exist two 6-dimensional subspaces V_1, V_2 in $\mathbb{R}^{10}$ whose intersection $V_1 \cap V_2$ is a line.

(i) There do not exist two 6-dimensional subspaces V_1, V_2 in $\mathbb{R}^{10}$ whose intersection $V_1 \cap V_2$ is a 5-dimensional subspace.

(j) For a field F, any matrix A in $F^{n \times n}$ satisfying $A^3 - I_{n \times n} = 0$ is invertible.

(k) For a field F, any matrix A in $F^{n \times n}$ satisfying $A^4 - A = 0$ is noninvertible.

2. (20 points total) Let H, K be subgroups of a finite group G, and define as usual the subset $HK := \{hk : h \in H, k \in K\} \subseteq G$.

(a) (5 points) Give an example of H, K, G where HK is not a subgroup of G.

(b) (5 points) Prove that the set map $H \times K \xrightarrow{\varphi} G$ defined by $\varphi((h,k) := hk$ has this property: for every g in HK, the fiber $\varphi^{-1}(g) := \{(h,k) \in H \times K : \varphi((h,k)) = g\}$ has the same cardinality $|\varphi^{-1}(g)|$.

(c) (10 points) Use part (b) to prove that $|HK| = \frac{|H||K|}{|H \cap K|}$.

3. (15 points total) Recall from Artin's Exercise 7.10.7 that for a group G, the *commutator subgroup* $C = \langle \{ghg^{-1}h^{-1}\}_{g,h \in G} \rangle$ is the subgroup of G generated by all commutators $ghg^{-1}h^{-1}$, that $C \triangleleft G$ is a normal subgroup, and that the quotient group G/C is abelian.

(a) (5 points) Prove that when G is the symmetric group S_n with $n \geq 2$, the canonical quotient map $G \xrightarrow{\pi} G/C$ sends every tranposition $t = (i,j)$ to the same element of G/C, that is $\pi(t) = \pi(t')$ for all transpositions t, t' in S_n.

(b) (10 points) Prove that $G = S_n$ for $n \geq 2$ has $G/C \cong \{\pm 1\}$.

4. (20 points total, 10 points each) Letting n be an integer with $n \geq 2$ and $\zeta := e^{\frac{2\pi i}{2n}}$ in $\mathbb{C}$, define $G := \langle a, b \rangle$ as the subgroup of $GL_2(\mathbb{C})$ generated by these two matrices:

$$a = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \text{ and } b = \begin{bmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{bmatrix}.$$

(a) (10 points) Prove that $|G| = 4n$.

(b) (10 points) Prove that G has this presentation by generators and relations:

$$G \cong \langle\, \alpha, \beta \mid \alpha^4 = 1 = \beta^{2n},\ \beta^n = \alpha^2,\ \alpha\beta\alpha^{-1} = \beta^{-1} \rangle.$$