Math 5285 Honors abstract algebra
Spring 2008, Vic Reiner
Final exam - Due Friday May 9, in class

Instructions: This is an open book, open library, open notes, open web, take-home exam, but you are *not* allowed to collaborate. The instructor is the only human source you are allowed to consult.

1. (30 points total; 5 points each part)

(a) (5 points) Show that in the symmetric group S_n, conjugating an m-cycle gives an m-cycle, and more specifically

$$\sigma(a_1\, a_2\, \cdots\, a_{m-1}\, a_m)\sigma^{-1} = (\sigma(a_1)\,\sigma(a_2)\,\cdots\sigma(a_{m-1})\,\sigma(a_m)).$$

(b) (5 points) Show that in the symmetric group S_5, the subgroup $\langle \tau, \sigma \rangle$ generated by *any* 2-cycle $\tau = (ij)$ together with *any* 5-cycle $(a\,b\,c\,d\,e)$ is the whole group S_5.

(Recall that we wanted this fact from (b) in lecture, in order to conclude that an irreducible quintic polynomial $f(x) \in \mathbb{Q}[x]$ that had exactly 3 real roots in $\mathbb{C}$ has Galois group $G(\mathrm{Split}_{\mathbb{Q}}(f)/\mathbb{Q})$ isomorphic to S_5.)

(c) (5 points) Which cycle types (= lists of cycle sizes) for permutations of S_5 are the ones that lie in the subgroup of *alternating* permutations A_5?

(d) (5 points) Show that the two 5-cycles (12345) and (21345) are conjugate within S_5, but *not* conjugate within A_5.

(e) (5 points) Write down the class equation for A_5, that is, the list of sizes of all of the conjugacy classes, and how they add up to $|A_5|$.

(f) (5 points) Prove that a *normal* subgroup H of a finite group G must have its cardinality $|H|$ expressible as a sum of cardinalities of distinct conjugacy classes in G, and one of these cardinalities must be 1, corresponding to the identity conjugacy class $\{e\}$.

Use this to deduce that A_5 is a simple group (i.e. it has no non-identity proper normal subgroups), and hence is *not* a solvable group. Explain why this proves S_5 is also *not* a solvable group.

(Recall that we wanted this to conclude that the quintic polynomials $f \in \mathbb{Q}[x]$ with exactly 3 real roots mentioned above are not solvable by radicals).

2. (15 points total) Let $(\mathbb{Q} \subset)\mathbb{F} \subset \mathbb{K}$ be a Galois extension in characteristic zero, with Galois group $G(\mathbb{K}/\mathbb{F}) \cong D_4$, the dihedral group of order 8, the symmetries of a square.

(a) (10 points) How many intermediate subfields $\mathbb{L}$ are there lying strictly between $\mathbb{F}$ and $\mathbb{K}$, that is, with $\mathbb{F} \subsetneq \mathbb{L} \subsetneq \mathbb{K}$?

(b) (5 points) How many of the intermediate subfields $\mathbb{L}$ from part (a) have $\mathbb{L}/\mathbb{F}$ Galois?

3. (20 points total) Consider the matrix $A \in \mathbb{Z}^{4\times 4}$ shown below

$$A = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}$$

as representing a $\mathbb{Z}$-module homomorphism $\mathbb{Z}^4 \xrightarrow{A} \mathbb{Z}^4$ with respect to the standard basis for $\mathbb{Z}^4$ in both the domain and range.

Write the two finitely generated $\mathbb{Z}$-modules $\ker A$ and $\mathbb{Z}^4/\operatorname{im} A$ explicitly in the form

$$\mathbb{Z} \oplus \cdots \oplus \mathbb{Z} \oplus \mathbb{Z}/n_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/n_r\mathbb{Z}$$

guaranteed by the theorem on finitely generated modules over a Euclidean domain.

4. (15 points total) Artin's Problem 12.6.4 on page 487.

5. (10 points total) Artin's Problem 12.7.21 on page 489.

6. (10 points total) Consider the matrix $A \in \mathbb{C}^{5\times 5}$ shown below

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

making $V = \mathbb{C}^5$ a finitely-generated $\mathbb{C}[t]$-module, in which t acts on an element v in $V = \mathbb{C}^5$ as left-multiplication by A.

Write V explicitly in the form

$$\mathbb{C}[t] \oplus \cdots \oplus \mathbb{C}[t] \oplus \mathbb{C}[t]/(f_1(t)) \oplus \cdots \oplus \mathbb{C}[t]/(f_r(t))$$

guaranteed by the theorem on finitely generated modules over a Euclidean domain.