Math 5285 Honors abstract algebra
Spring 2008, Vic Reiner
Midterm exam 1- Due Wednesday March 5, in class

Instructions: This is an open book, open library, open notes, open web, take-home exam, but you are *not* allowed to collaborate. The instructor is the only human source you are allowed to consult.

1. (30 points total; 10 points each part) Define three rings by adjoining roots of quartic polynomials to $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$:

$$R_1 = \mathbb{F}_2[x]/(x^4 + x + 1)$$
$$R_2 = \mathbb{F}_2[x]/(x^4 + x^2 + 1)$$
$$R_3 = \mathbb{F}_2[x]/(x^4 + x^3 + 1)$$

(a) Prove that R_1, R_3 are fields, but that R_2 is not a field.
(b) Find a *primitive element* in R_1, that is, an element β such that $R_1^\times = R_1 - \{0\} = \{1, \beta, \beta^2, \ldots\}$.
(c) Give an explicit ring isomorphism $R_1 \xrightarrow{\varphi} R_3$ (and *prove* that it is an isomorphism).

2. (15 points) Artin's Chapter 11 Section 2 Problem 2 part (b) on p. 442.

3. (20 points) Let $f(x) = ax^2 + bx + c$ in $\mathbb{R}[x]$ be *any* irreducible quadratic polynomial, that is, one with $b^2 - 4ac < 0$. Prove there is a ring isomorphism $\mathbb{R}[x]/(f(x)) \cong \mathbb{C}$.

4. (15 points) A field $\mathbb{F}$ is *algebraically closed* if every nonconstant polynomial $f(x)$ in $\mathbb{F}[x]$ has at least one root in $\mathbb{F}$. Prove that *finite* fields $\mathbb{F}$ are never algebraically closed.
(Hint: Given a finite field $\mathbb{F}$, can you write down an *explicit* nonconstant polynomial $f(x)$ that has no roots in $\mathbb{F}$?)

5. (20 points total; 10 points each) Two problems from Artin on rings of formal power series:
(a) Artin's Chapter 10 Section 2 Problem 6 part (b) on p. 380.
(b) Artin's Chapter 10 Section 3 Problem 26 on p. 382.