Math 5286 Honors fundamental structures of algebra– 2nd semester
Spring 2019, Vic Reiner
Midterm exam 2- Due Wednesday April 17, in class

Instructions: There are 4 problems. This is an open book, open library, open notes, open web, take-home exam, but you are *not* allowed to collaborate. The instructor is the only human source you are allowed to consult.

1. (40 points total, 5 points each part) True or False?
True assertions must be proven, and false assertions must be disproven.

(a) For a prime p, any irreducible $f(x)$ in $\mathbb{F}_p[x]$ that has at least one root in $\mathbb{F}_{p^d}$ will split completely in $\mathbb{F}_{p^d}[x]$, that is, $f(x) = c\prod_{i=1}^{n}(x-\alpha_i)$ with α_i in $\mathbb{F}_{p^d}$.

(b) The polynomial $f(x) = x^{33} + x^{32} - x^2 - x$ splits completely in $\mathbb{F}_{2^{701}}[x]$.

(c) The finite field $\mathbb{F}_{37^2}$ is a splitting field over $\mathbb{F}_{37}$ for $f(x) = x^2 - 1$ in $\mathbb{F}_{37}[x]$.

(d) The finite field $\mathbb{F}_{2^{701}}$ contains $\mathbb{F}_{2^5}$ as a subfield.

(e) For any positive integers d, e, the compositum $\mathbb{K}_1\mathbb{K}_2$ within $\mathbb{R}$ of the two subfields $\mathbb{K}_1 = \mathbb{Q}(\sqrt[d]{5})$ and $\mathbb{K}_2 = \mathbb{Q}(\sqrt[e]{5})$ is $\mathbb{Q}(\sqrt[de]{5})$

(f) There exists a field isomorphism $\mathbb{Q}(\sqrt[2]{5}) \cong \mathbb{Q}(\sqrt[2]{11})$.

(g) Every field extension $\mathbb{K}/\mathbb{Q}$ with $[\mathbb{K} : \mathbb{Q}] = 2$ is Galois.

(h) Every field extension $\mathbb{K}/\mathbb{Q}$ with $[\mathbb{K} : \mathbb{Q}] = 4$ is Galois.

2. (20 points total; 10 points each part)

(a) (10 points) Prove that $x^{15} - 17$ is irreducible in $\mathbb{Q}[x]$.

(b) (10 points) Prove that $x^5 - \sqrt[3]{17}$ is irreducible in $\mathbb{Q}(\sqrt[3]{17})[x]$.

3. (20 points total; 10 points each part) Let p be a prime number.

(a) Give a formula, as a function of p, for the number of monic irreducible polynomials $f(x)$ in $\mathbb{F}_p[x]$ of degree 5.

(b) Give a formula, as a function of p, for the number of monic irreducible polynomials $f(x)$ in $\mathbb{F}_p[x]$ of degree 15.

4. (20 points total; 5 points each part) Let $\alpha = +\sqrt[4]{7}$ be the positive real fourth root of 7, and let $\mathbb{K} = \mathbb{Q}(\alpha)$.

(a) What is $[\mathbb{K} : \mathbb{Q}]$? Prove your answer.

(b) Prove or disprove: there exists σ in $G(\mathbb{K}/\mathbb{Q})$ with $\sigma(\sqrt{7}) = -\sqrt{7}$.

(c) Describe the group $G(\mathbb{K}/\mathbb{Q})$ explicitly in this case.

(d) Among all extensions $\mathbb{K}'$ of $\mathbb{K}$ for which $\mathbb{K}'/\mathbb{Q}$ is Galois, what is the smallest possible extension degree $[\mathbb{K}' : \mathbb{Q}]$? Prove your answer.