Name: VIC REINER - ANSWER KEY

Signature: ______________________

Math 5651 (V. Reiner) Final Exam
Tuesday, May 10, 2016

This is a 120 minute exam. No books, notes, calculators, cell phones, watches or other electronic devices are allowed. You can leave answers as fractions, with binomial or multinomial coefficients, and Gamma function values $\Gamma(\alpha)$ unevaluated.

There are a total of 100 points. To get full credit for a problem you must show the details of your work. Answers unsupported by an argument will get little credit. Do all of your calculations on this test paper.

Problem	Score
1.	
2.	
3.	
4.	
5.	
Total:	

Reminders:

$$\mathbf{Pr}(A_1 \cup \cdots \cup A_n) = \sum_{k=1}^{n} (-1)^{k-1} \sum_{1 \leq i_1 < \cdots < i_k \leq n} \mathbf{Pr}(A_{i_1} \cap \cdots \cap A_{i_k})$$

$$S = \sqcup_{i=1}^{n} B_i \Rightarrow \mathbf{Pr}(A) = \sum_{i=1}^{n} \mathbf{Pr}(A \cap B_i) = \sum_{i=1}^{n} \mathbf{Pr}(A|B_i)\mathbf{Pr}(B_i) \text{ and } \mathbf{Pr}(B_i|A) = \mathbf{Pr}(A|B_i)\mathbf{Pr}(B_i)/\mathbf{Pr}(A)$$

$$\text{cdf } F(x) := \mathbf{Pr}(X \leq x), \text{while pdf } f(x) = \frac{\partial}{\partial x} F(x), \text{ and } g_1(x|y) = f(x,y)/f_2(y) \text{ with } f_2(y) = \int_{x=-\infty}^{x=+\infty} f(x,y)dx$$

$$\text{When } \underline{Y} = \underline{r}(\underline{X}) \Leftrightarrow \underline{X} = \underline{s}(\underline{Y}), \text{ then } f(\underline{x}), g(\underline{y}) \text{ satisfy } g(\underline{y}) = f(\underline{s}(y)) \cdot |J| \text{ where } J := \det\left(\frac{\partial s_i}{\partial y_j}\right)$$

$$\mathbf{E}X = \int_{-\infty}^{+\infty} x f(x)dx, \text{ and } \mathrm{var}(X) = E(X-EX)^2 = E(X^2) - (EX)^2, \text{ with } \sigma(X) := +\sqrt{\mathrm{var}(X)}$$

$$\mathrm{cov}(X,Y) = E((X-EX)(Y-EY)) = E(XY) - EX \cdot EY = \sigma_X \sigma_Y \rho(X,Y)$$

$$\mathrm{var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \mathrm{var}(X_i) + 2 \sum_{1 \leq i < j \leq n} \mathrm{cov}(X_i, X_j)$$

$$\Gamma(\alpha) := \int_{x=0}^{x=\infty} x^{\alpha-1} e^{-x} dx \text{ for } \alpha > 0, \text{ and } \Gamma(\alpha+1) = \alpha\Gamma(\alpha) \text{ for } \alpha > 1, \text{ with } \Gamma(1) = 1$$

Discrete Distributions

	Bernoulli with parameter p	Binomial with parameters n and p
p.f.	$f(x) = p^x(1-p)^{1-x}$, for $x = 0, 1$	$f(x) = \binom{n}{x} p^x (1-p)^{n-x}$, for $x = 0, \ldots, n$
Mean	p	np
Variance	$p(1-p)$	$np(1-p)$
m.g.f.	$\psi(t) = pe^t + 1 - p$	$\psi(t) = (pe^t + 1 - p)^n$

	Uniform on the integers $a, \ldots, b$	Hypergeometric with parameters A, B, and n
p.f.	$f(x) = \frac{1}{b-a+1}$, for $x = a, \ldots, b$	$f(x) = \frac{\binom{A}{x}\binom{B}{n-x}}{\binom{A+B}{n}}$, for $x = \max\{0, n-b\}, \ldots, \min\{n, A\}$
Mean	$\frac{b+a}{2}$	$\frac{nA}{A+B}$
Variance	$\frac{(b-a)(b-a+2)}{12}$	$\frac{nAB}{(A+B)^2}\frac{A+B-n}{A+B-1}$
m.g.f.	$\psi(t) = \frac{e^{(b+1)t} - e^{at}}{(e^t - 1)(b-a+1)}$	Nothing simpler than $\psi(t) = \sum_x f(x)e^{tx}$

	Geometric with parameter p	Negative binomial with parameters r and p
p.f.	$f(x) = p(1-p)^x$, for $x = 0, 1, \ldots$	$f(x) = \binom{r+x-1}{x} p^r (1-p)^x$, for $x = 0, 1, \ldots$
Mean	$\frac{1-p}{p}$	$\frac{r(1-p)}{p}$
Variance	$\frac{1-p}{p^2}$	$\frac{r(1-p)}{p^2}$
m.g.f.	$\psi(t) = \frac{p}{1-(1-p)e^t}$, for $t < \log(1/[1-p])$	$\psi(t) = \left(\frac{p}{1-(1-p)e^t}\right)^r$, for $t < \log(1/[1-p])$

	Poisson with mean λ	Multinomial with parameters n and $(p_1, \ldots, p_k)$
p.f.	$f(x) = e^{-\lambda}\frac{\lambda^x}{x!}$, for $x = 0, 1, \ldots$	$f(x_1, \ldots, x_k) = \binom{n}{x_1, \ldots, x_k} p_1^{x_1} \cdots p_k^{x_k}$, for $x_1 + \cdots + x_k = n$ and all $x_i \geq 0$
Mean	λ	$E(X_i) = np_i$, for $i = 1, \ldots, k$
Variance	λ	$\mathrm{Var}(X_i) = np_i(1-p_i)$, $\mathrm{Cov}(X_i, X_j) = -np_ip_j$, for $i, j = 1, \ldots, k$
m.g.f.	$\psi(t) = e^{\lambda(e^t - 1)}$	Multivariate m.g.f. can be defined, but is not defined in this text.

Continuous Distributions

	Beta with parameters α and β	Uniform on the interval $[a, b]$
p.d.f.	$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$, for $0 < x < 1$	$f(x) = \frac{1}{b-a}$, for $a < x < b$
Mean	$\frac{\alpha}{\alpha+\beta}$	$\frac{a+b}{2}$
Variance	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	$\frac{(b-a)^2}{12}$
m.g.f.	Not available in simple form	$\psi(t) = \frac{e^{-at} - e^{-bt}}{t(b-a)}$

	Exponential with parameter β	Gamma with parameters α and β
p.d.f.	$f(x) = \beta e^{-\beta x}$, for $x > 0$	$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$, for $x > 0$
Mean	$\frac{1}{\beta}$	$\frac{\alpha}{\beta}$
Variance	$\frac{1}{\beta^2}$	$\frac{\alpha}{\beta^2}$
m.g.f.	$\psi(t) = \frac{\beta}{\beta - t}$, for $t < \beta$	$\psi(t) = \left(\frac{\beta}{\beta - t}\right)^\alpha$, for $t < \beta$

	Normal with mean μ and variance σ^2	Bivariate normal with means μ_1 and μ_2, variances σ_1^2 and σ_2^2, and correlation ρ
p.d.f.	$f(x) = \frac{1}{(2\pi)^{1/2}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$	Formula is too large to print here. See Eq. (5.10.2) on page 338.
Mean	μ	$E(X_i) = \mu_i$, for $i = 1, 2$
Variance	σ^2	Covariance matrix: $\begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$
m.g.f.	$\psi(t) = \exp\left(\mu t + \frac{t^2\sigma^2}{2}\right)$	Bivariate m.g.f. can be defined, but is not defined in this text.

Problem 1. *(20 points)* Assume that each of the functions $f(k)$ or $f(x)$ below defines a p.f. or p.d.f., and find the unknown constant c.

a. (4 points) $f(k) = c\frac{(6.3)^k}{k!}$ for $k = 0, 1, 2, \ldots$

$\mathrm{Poi}(\underset{\lambda}{6.3})$ has ~~pf~~ $e^{-6.3}\frac{(6.3)^k}{k!}$ for $k=0,1,2,\ldots$ $\Rightarrow$ $c = e^{-6.3}$

1/4 for in any part for only saying $1 = \int_{x\in\mathbb{R}} c\,g(x)\,dx$ so $c = \frac{1}{\int_{x\in\mathbb{R}} g(x)\,dx}$

b. (4 points) $f(x) = \begin{cases} cx^{5.3}(1-x)^{7.14} & \text{for } x \in (0,1), \\ 0 & \text{otherwise.} \end{cases}$

$\mathrm{Beta}(\underset{\alpha}{6.3}, \underset{\beta}{8.14})$ has p.d.f. $\begin{cases} \frac{\Gamma(6.3+8.14)}{\Gamma(6.3)\Gamma(8.14)} x^{5.3}(1-x)^{7.14} & \text{for } x\in(0,1) \\ 0 & \text{otherwise} \end{cases}$ $\Rightarrow$ $c = \frac{\Gamma(\overbrace{6.3+8.14}^{14.44})}{\Gamma(6.3)\Gamma(8.14)}$

−1/4 for $\alpha=5.3$ or $\beta=7.14$

c. (4 points) $f(k) = c\binom{40}{k}\binom{50}{13-k}$ for $k = 0, 1, 2, \ldots, 13$.

$\mathrm{Hypergeom}(\underset{A}{40}, \underset{B}{50}, \underset{n}{13})$ has p.f. $\frac{\binom{40}{k}\binom{50}{13-k}}{\binom{40+50}{13}}$ for $k=0,1,\ldots,13$ $\Rightarrow$ $c = \frac{1}{\binom{90}{13}}$

3/4 for $\binom{90}{13}$

d. (4 points) $f(x) = \begin{cases} ce^{-5.3x+2} & \text{for } x > 0, \\ 0 & \text{otherwise.} \end{cases}$

$\mathrm{Exp}(-5.3)$ has p.d.f. $\begin{cases} 5.3e^{-5.3x} & \text{for } x>0 \\ 0 & \text{otherwise} \end{cases}$ $\Rightarrow$ $5.3e^{-5.3x} = ce^{-5.3x+2}$

$= ce^{2}\cdot e^{-5.3x}$

$5.3 = ce^{2}$

$c = 5.3e^{-2}$

−1/4 for omitting e^2

e. (4 points) $f(x) = \begin{cases} ce^{-5.3x^2+2} & \text{for } x \in \mathbb{R}, \\ \cancel{0} & \cancel{\text{otherwise.}} \end{cases}$

$N(\mu,\sigma)$ has p.d.f. $\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$, so rewrite $ce^{-5.3x^2+2} = ce^{2}\cdot e^{-5.3x^2} = ce^{2}\underbrace{e^{-\frac{1}{2}\left(\frac{x-0}{1/\sqrt{10.6}}\right)^2}}_{\text{looks like } N(\mu,\sigma)}$

$\mu = 0$, $\sigma = 1/\sqrt{10.6}$

2/4

$\Rightarrow ce^{2} = \frac{1}{\sigma\sqrt{2\pi}} = \frac{1}{\frac{1}{\sqrt{10.6}}\sqrt{2\pi}}$

$c = e^{-2}\sqrt{\frac{5.3}{\pi}}$

−1/4 for arithmetic errors here

Problem 2. *(20 points)* Let $X = \text{Unif}(0,1)$ be uniform on $[0,1]$.

a. (5 points) For which parameters (α, β) is $X = \text{Unif}(0,1)$ the same as the Beta distribution with parameters (α, β)?

Beta(α,β) has pdf $\begin{cases} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} & \text{for } x\in(0,1) \\ 0 & \text{otherwise} \end{cases}$, so need $\begin{matrix}\alpha-1=0\\ \beta-1=0\end{matrix}$

to match the pdf. $\begin{cases} 1 & \text{for } x\in(0,1) \\ 0 & \text{otherwise}\end{cases}$ of $X=\text{Unif}(0,1)$. Hence $\alpha=1$, $\beta=1$.

b. (15 points) Given $\gamma > 0$ a positive real number, and $X = \text{Unif}(0,1)$ as before, prove that $Y = X^\gamma$ is Beta distributed with parameters $\alpha = \frac{1}{\gamma}, \beta = 1$, that is, $Y = \text{Beta}(\frac{1}{\gamma}, 1)$.

$$Y = X^{\gamma} = r(X) \iff X = Y^{\frac{1}{\gamma}} = s(Y)$$

5/15

monotone
$\begin{cases}\text{increasing if } 0<\gamma<1\\ \text{decreasing if } 1<\gamma\end{cases}$
for $x\in(0,1)$

8/15 for saying various correct things that don't get you there

(The case $\gamma=1$ is trivial, since $Y=X$ in that case)

$$\frac{ds}{dy} = \frac{1}{\gamma} y^{\frac{1}{\gamma}-1}$$

So Y has pdf $g(y) = \begin{cases} f(s(y))\cdot\left|\frac{ds}{dy}\right| & \text{for } y\in(0,1) \\ 0 & \text{otherwise}\end{cases}$ where $f(x) = \begin{cases} 1 & \text{for } x\in(0,1) \\ 0 & \text{otherwise}\end{cases}$ since $X=\text{Unif}(0,1)$

$$= \begin{cases} \left|\frac{1}{\gamma} y^{\frac{1}{\gamma}-1}\right| = \frac{1}{\gamma} y^{\frac{1}{\gamma}-1} & \text{for } y\in(0,1) \\ 0 & \text{otherwise}\end{cases}.$$

This IS the same p.d.f. as Beta$(\frac{1}{\gamma},1)$, which would be

$$\begin{cases} \frac{\Gamma(\frac{1}{\gamma}+1)}{\Gamma(\frac{1}{\gamma})\Gamma(1)} y^{\frac{1}{\gamma}-1}(1-y)^{1-1} = \frac{\Gamma(\frac{1}{\gamma}+1)}{\Gamma(\frac{1}{\gamma})}\cdot y^{\frac{1}{\gamma}-1} = \frac{1}{\gamma}y^{\frac{1}{\gamma}-1} & \text{for } y\in(0,1), \\ 0 & \text{otherwise.}\end{cases}$$

3/15

7/15 for recognizing Beta$(\frac{1}{\gamma},1)$ has pdf $\frac{1}{\gamma}y^{\frac{1}{\gamma}-1} = \frac{d}{dy}y^{\frac{1}{\gamma}}$

Problem 3. *(20 points)* A group of 20 people $\{P_1, Q_1, P_2, Q_2, \dots, P_{10}, Q_{10}\}$ who happen to be 10 pairs $\{P_i, Q_i\}$ of twins are going swimming, and randomly pair off as 10 pairs of swimming buddies. Assume all possible pairings of the 20 people are equally likely. In particular, two twins $\{P_i, Q_i\}$ may or may not pair with each other.

Let X be the number of pairs of twins which end up as swim buddies, that is, X is the number of indices i for which P_i, Q_i are paired with each other. Thus X takes values in $\{0, 1, 2, \dots, 10\}$.

a. (5 points) Find the probability that P_1, Q_1 are paired with each other.

$$\Pr(\underbrace{P_1, Q_1 \text{ paired}}_{\text{event } A_1 :=}) = \frac{|A_1|}{|S|} \quad \text{where } S = \{\text{all possible pairings}\}$$

$$= \frac{(20-3)\cdots 5\cdot 3\cdot 1}{(20-1)(20-3)\cdots 5\cdot 3\cdot 1} = \frac{1}{19}$$

3/5 for $\dfrac{\binom{18}{2,3,3,3,4,4,3,3,2}}{\binom{20}{2,3,3,3,3,3,3,3,2}}$

1/5 for $\frac{9!}{10!} = \frac{1}{10}$

b. (5 points) Compute the expected value $\mathbf{E}X$.

$$X = X_1 + X_2 + \dots + X_{10} \quad \text{where } X_i = \begin{cases} 1 & \text{if } \overbrace{P_i, Q_i \text{ paired}}^{\text{event } A_i :=} \\ 0 & \text{else} \end{cases}$$

$$\text{so } EX = EX_1 + EX_2 + \dots + EX_{10} = 10\,EX_1 = 10\Pr(A_1) = 10 \cdot \frac{1}{19} = \frac{10}{19}$$

2/5

4/5 for $\frac{20}{19}$

4/5 for using $EX = pn = \frac{10}{19}$ for $Bin(10, \frac{1}{19})$

1/5 for Hypergeom$(A, B, 10)$ with some A, B

c. (5 points) Find the probability that *both* P_1, Q_1 are paired with each other *and* P_2, Q_2 are paired with each other.

$$\Pr(A_1 \cap A_2) = \frac{|A_1 \cap A_2|}{|S|} = \frac{(20-5)\cdots 5\cdot 3\cdot 1}{(20-1)(20-3)(20-5)\cdots 5\cdot 3\cdot 1} = \frac{1}{19\cdot 17}$$

1/5 for $\frac{8!}{10!} = \frac{1}{9\cdot 10}$

d. (5 points) Compute the variance $\mathrm{Var}(X)$.

$$\mathrm{Var}(X) = \mathrm{var}(X_1 + \dots + X_{10}) \overset{2/5}{=} \sum_{i=1}^{10} \mathrm{var}(X_i) + 2\sum_{1\le i<j\le 10}' \mathrm{cov}(X_i, X_j)$$

$$\overset{1/5}{=} 10\,\mathrm{var}(X_1) + 2\binom{10}{2}\mathrm{cov}(X_1, X_2)$$

$$\mathrm{var}(X_1) = E(X_1^2) - E(X_1)^2 = E(X_1) - E(X_1)^2 = \frac{1}{19} - \frac{1}{19^2}$$

$$\mathrm{cov}(X_1, X_2) = E(X_1X_2) - E(X_1)E(X_2) = \Pr(A_1 \cap A_2) - E(X_1)^2 = \frac{1}{19\cdot 17} - \frac{1}{19^2}$$

$$\Rightarrow \mathrm{Var}(X) = 10\left(\frac{1}{19} - \frac{1}{19^2}\right) + 2\binom{10}{2}\left(\frac{1}{19\cdot 17} - \frac{1}{19^2}\right)$$

$$= \frac{9\cdot 10\cdot 36}{17\cdot 19^2}$$

3/5 for trying to use $\mathrm{var}(X) = E(X^2) - (EX)^2$ incorrectly

Problem 4. *(20 points)* Let (X_1, X_2) have a bivariate normal distribution with means $\mathbf{E}X_1 = 10, \mathbf{E}X_2 = 8$, standard deviations $\sigma(X_1) = 1, \sigma(X_2) = 3$, and correlation $\rho(X_1, X_2) = 0.5$. Let $Y = X_1 - X_2$.

a. (5 points) What choice of an estimate m_0 for Y will minimize the mean square error $\mathbf{E}((Y - m_0)^2)$?

$$m_0 = EY = E(X_1 - X_2) = EX_1 - EX_2 = 10 - 8 = 2.$$

2/5 2/5

b. (5 points) What choice of an estimate m_1 for Y will minimize the mean absolute error $\mathbf{E}(|Y - m_1|)$? Since (X_1, X_2) are bivariate normal, $Y = X_1 - X_2$ is (univariate) normal, and its $\boxed{\text{median } m_1 \text{ (what we want)}}$ is the same as its mean EY – its pdf is symmetric about the mean. Hence $m_1 = m_0 = EY = 2$.

-2/5

3/5 for trying to evaluate $\frac{1}{2} = \int_{-\infty}^{m_1} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$

c. (5 points) Choosing m_0 as in part (a), compute $\mathbf{E}((Y - m_0)^2)$.

$$E((Y-m_0)^2) = E((Y-EY)^2) = var(Y) \,(= \sigma^2(Y)) = var(X_1 - X_2)$$

2/5

$$= var(X_1) + var(X_2) - 2\,cov(X_1, X_2)$$

2/5 $\quad -\frac{1}{5}$ for $+$

$$= \sigma(X_1)^2 + \sigma(X_2)^2 - 2\rho(X_1,X_2)\sigma(X_1)\sigma(X_2)$$

$$= 1^2 + 3^2 - 2\,(0.5)\,1 \cdot 3$$

$$= 1 + 9 - 3 = 7$$

1/5

3/5 for $var(Y) = var(X_1) + var(X_2)$

d. (5 points) Express $\mathbf{Pr}(X_1 > X_2)$ in the terms of the cdf $\Phi(z) = \int_{-\infty}^{z} e^{\frac{t^2}{2}}\,dt$ for a standard normal $Z = \mathcal{N}(0, 1)$ having mean 0, standard deviation 1.

$$Pr(X_1 > X_2) = Pr(\underbrace{Y}_{N(\underset{\mu}{2},\, \underset{\sigma}{\sqrt{7}})} > 0) = Pr\Big(\underbrace{\frac{Y-2}{\sqrt{7}}}_{Z = N(0,1)} > \frac{-2}{\sqrt{7}}\Big) = 1 - Pr\Big(Z \le \frac{-2}{\sqrt{7}}\Big)$$

$$= 1 - \Phi\Big(\frac{-2}{\sqrt{7}}\Big)$$

1/5 1/5

3/5 for using wrong μ

Problem 5. *(20 points)* Let X be a Poisson random variable with unknown mean λ, and you have been told, *a priori*, that λ comes from a Gamma distribution Gamma$(2,3)$ with parameters $\alpha = 2, \beta = 3$.

You sample X and obtain the value $X = 5$.

Show that the *posterior* distribution for λ still follows a Gamma distribution Gamma$(\hat{\alpha}, \hat{\beta})$, and find the new values $(\hat{\alpha}, \hat{\beta})$ explicitly.

5/20

$$(X, \Lambda) \qquad \Lambda = \text{Gamma}(\overset{\alpha}{2}, \overset{\beta}{3}) \Rightarrow f_2(\lambda) = \begin{cases} \frac{3^2}{\Gamma(2)} \lambda^{2-1} e^{-3\lambda} & \text{for } \lambda > 0 \\ 0 & \text{otherwise.} \end{cases}$$

$$X = \text{Poi}(\lambda) \Rightarrow g_1(x|\lambda) = e^{-\lambda} \frac{\lambda^x}{x!} \quad \text{for } x = 0,1,2,\ldots$$

5/20

Therefore ~~the~~ joint p.d.f. for (X, Λ) is

$$f(x,\lambda) = g_1(x|\lambda) f_2(\lambda) = \begin{cases} \frac{3^2}{\Gamma(2)\cdot x!} e^{-\lambda - 3\lambda} \lambda^{x+2-1} & \text{for } \lambda > 0, \; x = 0,1,2,\ldots \\ 0 & \text{otherwise} \end{cases}$$

We want the conditional p.d.f.

$$g_2(\lambda | x=5) = \frac{f(5,\lambda)}{f_1(5)} = \begin{cases} \frac{3^2}{\Gamma(2) \cdot 5! \cdot f_1(5)} \lambda^{5+2-1} e^{-\lambda-3\lambda} & \text{for } \lambda > 0 \\ 0 & \text{otherwise} \end{cases}$$

5/20

13/20 for getting confused about x versus λ e.g. $\frac{3^2}{\Gamma(2)} x^{2-1} e^{-3\lambda}$

$$= \begin{cases} (\text{some constant}) \underbrace{\lambda^{7-1} e^{-4\lambda}}_{\text{same as Gamma}(7,4)\text{, up to a constant}} & \text{for } \lambda > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\Rightarrow g_2(\lambda | x=5) \text{ is the pdf for Gamma}(\underset{\hat{\alpha}}{7}, \underset{\hat{\beta}}{4})$$

5/20

18/20 for Gamma(5,4)

15/20 for Gamma(7,-4)