Name: ______________________________

Signature: ______________________________

Math 5651 Lecture 003 (V. Reiner) Midterm Exam I
Thursday, February 22, 2018

This is a 115 minute exam. No books, notes, calculators, cell phones, watches or other electronic devices are allowed. You can leave answers as fractions, with binomial or multinomial coefficients unevaluated.

There are a total of 100 points. To get full credit for a problem you must show the details of your work. Answers unsupported by an argument will get little credit. Do all of your calculations on this test paper.

Problem	Score
1.	________
2.	________
3.	________
4.	________
5.	________
6.	________
Total:	________

Reminders:

$$\mathbf{Pr}(A_1 \cup \cdots \cup A_n) = \sum_{k=1}^{n} (-1)^{k-1} \sum_{1 \leq i_1 < \cdots < i_k \leq n} \mathbf{Pr}(A_{i_1} \cap \cdots \cap A_{i_k})$$

$$S = B_1 \sqcup \cdots \sqcup B_n \Rightarrow \mathbf{Pr}(A) = \sum_{i=1}^{n} \mathbf{Pr}(A \cap B_i) = \sum_{i=1}^{n} \mathbf{Pr}(A|B_i)\mathbf{Pr}(B_i),$$

and Bayes' Theorem: $\mathbf{Pr}(B_i|A) = \mathbf{Pr}(A|B_i)\mathbf{Pr}(B_i)/\mathbf{Pr}(A)$

$$X = \text{Bin}(n,p) \text{ has p.f. } f(k) = \binom{n}{k} p^k (1-p)^{n-k} \text{ for } k \in \{0,1,2,\ldots,n\}$$

$$X = \text{Hypergeom}(A,B,n) \text{ has p.f. } f(k) = \binom{A}{k}\binom{B}{n-k} \Big/ \binom{A+B}{n} \text{ for } k \in \{0,1,2,\ldots,\min\{A,n\}\}$$

$$X = \text{Poi}(\lambda) \text{ has p.f } f(k) = e^{-\lambda}\frac{\lambda^k}{k!} \text{ for } k \in \{0,1,2,\ldots\}$$

Problem 1. *(10 points)*

a. (3 points) You flip a fair coin 5 times. What is the probability that you never see two of the same flip consecutively, that is, no "heads, heads" and no "tails, tails"?

b. (7 points) Same question, except that this time you flip the coin n times, where $n \geq 2$. (Your answer for the probability should be a function of n.)

Problem 2. *(15 points)* You know that the number of house fires in a certain area during a month is a Poisson random variable with parameter λ.

a. (5 points) What is the probability that next month there are exactly 4 fires? (The parameter λ will appear in your answer.)

b. (10 points) If you know that last month there was *at least one* fire, then what is the probability that there were between 3-5 fires?

Problem 3. *(15 points)* Your favorite smoothie shop chooses its daily manager randomly among three possibilities: Huey and Dewey, who each work three days per week on average, and Louie who works one day per week on average. Huey and Dewey choose to make the tofu smoothie the daily special $\frac{1}{4}$ of the days that they work, and some other flavor the other $\frac{3}{4}$, while Louie *always* chooses tofu to be the daily special.

a. (5 points) What is the *a priori* chance that tomorrow Huey will be the manager?

b. (10 points) If you walk into the shop and the daily special smoothie is tofu, what is the chance that Louie is the manager today?

Problem 4. *(20 points total)*
True or False? Each of your answers **must be justified by calculation**.

a. (5 points) Given events A, B with $\mathbf{Pr}(A), \mathbf{Pr}(B) > 0$, if $A \cap B = \varnothing$, then A, B are dependent.

b. (5 points) Given an event A with $0 < \mathbf{Pr}(A) < 1$, and if we let $B = A$, then A, B are independent.

c. (5 points) When sampling **without replacement** $r > 0$ red and $w > 0$ white balls from a box, let R_i, W_i, respectively, be the events that the i^{th} ball sampled is red, white, respectively.

Then $\mathbf{Pr}(R_2|W_1) > \mathbf{Pr}(R_2) > \mathbf{Pr}(R_2|R_1)$.

d. (5 points) With notation as in part (c), events W_1 and R_2 are independent.

Problem 5. *(20 points)* Let A, B, C be events with $\mathbf{Pr}(C) > 0$. Prove that $\mathbf{Pr}(A \cup B|C) = \mathbf{Pr}(A|C) + \mathbf{Pr}(B|C) - \mathbf{Pr}(A \cap B|C)$.

Problem 6. *(20 points total)* You have two coins in your pocket, one *fair* coin having probability of heads 1/2, and one *unfair* coin in which the probability of heads is 3/4. You pull one of the two out of your pocket, with either coin equally probable, and start flipping it, generating a sequence of heads and tails.

a. (10 points) What is the probability that the *first* tails is on the 6^{th} flip?

b. (10 points) After you pull it out of your pocket, you do one test flip, and get tails. What is the probability that you pulled the fair coin out?