NOTE: REVISION TO INSTRUCTIONS FOR PROBLEM 4(d)! :
"Write down the explicit integral, but do not evaluate it."

Name: Vic Reiner - ANSWER KEY

Signature: ______________________

Math 5651 Lecture 003 (V. Reiner) Midterm Exam II

4:40pm - 6:35pm in FordH 115 Thursday, March 31, 2016

This is a 115 minute exam. No books, notes, calculators, cell phones, watches or other electronic devices are allowed. You can leave answers as fractions, with binomial or multinomial coefficients unevaluated.

There are a total of 100 points. To get full credit for a problem you must show the details of your work. Answers unsupported by an argument will get little credit. Do all of your calculations on this test paper.

Problem	Score
1.	20/20
2.	15/15
3.	15/15
4.	20/20
5.	15/15
6.	15/15
Total:	100/100

Reminders:

$$\mathbf{Pr}(A_1 \cup \cdots \cup A_n) = \sum_{k=1}^{n} (-1)^{k-1} \sum_{1 \le i_1 < \cdots < i_k \le n} \mathbf{Pr}(A_{i_1} \cap \cdots \cap A_{i_k})$$

$$S = \sqcup_{i=1}^{n} B_i \Rightarrow \mathbf{Pr}(A) = \sum_{i=1}^{n} \mathbf{Pr}(A \cap B_i) = \sum_{i=1}^{n} \mathbf{Pr}(A|B_i)\mathbf{Pr}(B_i) \text{ and } \mathbf{Pr}(B_i|A) = \mathbf{Pr}(A|B_i)\mathbf{Pr}(B_i)/\mathbf{Pr}(A)$$

$$\text{cdf } F(x) := \mathbf{Pr}(X \le x), \text{while pdf } f(x) = \frac{\partial}{\partial x} F(x)$$

$$g_1(x|y) = f(x,y)/f_2(y), \quad g_2(y|x) = f(x,y)/f_1(x)$$

$$f_1(x) = \int_{y=-\infty}^{y=+\infty} f(x,y)dy, \quad f_2(y) = \int_{x=-\infty}^{x=+\infty} f(x,y)dx$$

When $\underline{Y} = \underline{r}(\underline{X}) \Leftrightarrow \underline{X} = \underline{s}(\underline{Y})$, then $f(\underline{x}), g(\underline{y})$ satisfy $g(\underline{y}) = f(\underline{s}(y)) \cdot |J|$ where $J := \det\left(\frac{\partial s_i}{\partial y_j}\right)$

$$\mathbf{E}X = \begin{cases} \sum_k k \cdot f(k) & X \text{ discrete,} \\ \int_{-\infty}^{+\infty} x f(x)dx & X \text{ continuous.} \end{cases}$$

X	p.f. $f(k)$	$\mathbf{E}X$
$\mathrm{Bin}(n,p)$	$\binom{n}{k}p^k(1-p)^{n-k}$ for $k \in \{0,1,\ldots,n\}$	pn
$\mathrm{Hypergeom}(A,B,n)$	$\binom{A}{k}\binom{B}{n-k}/\binom{A+B}{n}$ for $k \in \{0,1,\ldots,\min\{A,n\}\}$	$\frac{A}{A+B}n$
$\mathrm{Poi}(\lambda)$	$e^{-\lambda}\frac{\lambda^k}{k!}$ for $k \in \{0,1,2,\ldots\}$	λ

Problem 1. *(20 points)* Let X_1, X_2 be random variables with joint pdf

$$f(x_1, x_2) = \begin{cases} 2x_1 & \text{if } 0 < x_1 < 1 \text{ and } 0 < x_2 < 1, \\ 0 & \text{otherwise.} \end{cases}$$

a. (10 points) Are X_1, X_2 independent? You must justify your answer.

Yes, since $f(x_1,x_2) \overset{2\text{pts}}{=} f_1(x_1) f_2(x_2)$, where $f_1(x_1) = \begin{cases} 2x_1 & \text{if } 0<x_1<1, \\ 0 & \text{otherwise} \end{cases}$

for all $(x_1,x_2) \in \mathbb{R}^2$ $\qquad f_2(x_2) = \begin{cases} 1 & \text{if } 0<x_2<1, \\ 0 & \text{otherwis} \end{cases}$

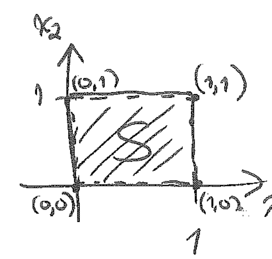
3pts

b. (10 points) Defining the random variable $Y := X_1 - X_2$, compute the pdf $g(y)$ for Y for all y in $\mathbb{R}$.

1/10 for subtracting pdfs of X_1, X_2

Let $Y_1 := Y = X_1 - X_2$ for $(x_1,x_2) \in (0,1)\times(0,1)$

$Y_2 = X_2$

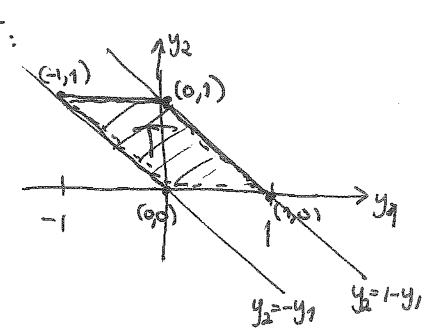


Then one has an invertible change-of-variables,

since $X_1 = Y_1 + Y_2 = s_1(Y_1,Y_2)$ for (y_1,y_2) in this region T:

$X_2 = Y_2 = s_2(Y_1,Y_2)$

3 pts



The Jacobian $J = \det\begin{bmatrix} \frac{\partial s_1}{\partial y_1} & \frac{\partial s_1}{\partial y_2} \\ \frac{\partial s_2}{\partial y_1} & \frac{\partial s_2}{\partial y_2} \end{bmatrix} = \det\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = 1$

and hence (Y_1, Y_2) have joint pdf

$g(y_1,y_2) = f(s_1(y_1,y_2), s_2(y_1,y_2)) \cdot |J| = \begin{cases} 2(y_1+y_2)\cdot 1 & \text{if } (y_1,y_2)\in T \\ 0 & \text{otherwise} \end{cases}$

3 more pts

1pt

Therefore $Y = Y_1$ has pdf $g(y) = g_1(y_1) = \int_{y_2=-\infty}^{y_2=+\infty} g(y_1,y_2)\, dy_2$

$$= \begin{cases} \int_{y_2=-y_1}^{y_2=1} 2(y_1+y_2)\,dy_2 = \left[2y_1y_2 + y_2^2\right]_{y_2=-y_1}^{y_2=1} = 2y_1+1-2y_1^2+y_1^2 = 1+2y_1-y_1^2 & \text{if } -1<y_1\le 0 \\ \int_{y_2=0}^{y_2=1-y_1} 2(y_1+y_2)\,dy_2 = \left[2y_1y_2+y_2^2\right]_{y_2=0}^{y_2=1-y_1} = 2y_1(1-y_1)+(1-y_1)^2 = (1-y_1)(2y_1+1-y_1) = 1-y_1^2 & \text{if } 0\le y_1<1 \\ 0 & \text{otherwise} \end{cases}$$

8/10 for not breaking into 3 pieces

3 more pts

Problem 2. *(15 points)* Assume that a 20 person committee is chosen from among 75 women and 25 men, with all possible choices equally likely. Let X denote the number of women on the committee, and Y the number of men on the committee.

a. (5 points) Calculate EX.

$$X = \text{Hypergeom}(\overset{A}{\overset{\shortparallel}{75}}, \overset{B}{\overset{\shortparallel}{25}}, \overset{n}{\overset{\shortparallel}{20}})$$

$$\text{so } EX = \frac{A}{A+B} n = \frac{75}{100} \cdot 20 = 15$$

4/5 for swapping A & B

b. (10 points) Calculate E$(X - Y)$.

$$Y = 20 - X$$

$$\text{so } E(X-Y) = E(X-(20-X))$$
$$= E(2X-20)$$
$$= 2E(X) - 20$$
$$= 2 \cdot 15 - 20$$
$$= 10$$

5/10 for right answer with no explanation

8/10 for swapping A & B

3/10 for linearity of expectation

Problem 3. *(15 points)* Let X be a discrete random variable whose values lie in $\{0, 1, 2, \ldots, n\}$. Prove that

$$EX = \Pr(X \geq 1) + \Pr(X \geq 2) + \cdots + \Pr(X \geq n-1) + \Pr(X \geq n)$$

$$\Pr(X \geq 1) + \Pr(X \geq 2) + \ldots + \Pr(X \geq n-1) + \Pr(X \geq n)$$

3 pts

$$= (f(1) + f(2) + \ldots + f(n)) +$$
$$(f(2) + \ldots + f(n)) +$$
$$\ldots +$$
$$+$$
$$(f(n-1) + f(n)) +$$
$$f(n)$$

$$= f(1) + 2f(2) + \ldots + (n-1)f(n-1) + nf(n)$$
$$= \sum_k k f(k)$$
$$\overset{\checkmark}{=} EX$$

3 pts

Problem 4. *(20 points)* Let X be a continuous random variable with pdf

$$f(x) = \begin{cases} c(x^2-4) & \text{if } -2 < x < 2, \\ 0 & \text{otherwise,} \end{cases}$$

where c is some constant lying in $\mathbb{R}$.

a. (5 points) Find the value of c.

$$1 = \int_{x=-\infty}^{x=+\infty} f(x)\,dx = \int_{x=-2}^{x=+2} c(x^2-4)\,dx = c\left[\frac{x^3}{3} - 4x\right]_{x=-2}^{x=+2} = c\left[\left(\frac{8}{3}-8\right) - \left(-\frac{8}{3}+8\right)\right] = c\left[\frac{16}{3} - 16\right]$$

$$= \frac{-32}{3}c \quad \Rightarrow c = \frac{-3}{32}$$

b. (5 points) Compute $\mathbf{E}X$.

$$\mathbf{E}X = \int_{x=-\infty}^{x=+\infty} x f(x)\,dx = \int_{x=-2}^{x=+2} \frac{-3}{32}\, x(x^2-4)\,dx = \frac{-3}{32}\int_{-2}^{+2} \underbrace{(x^3-4x)}_{\text{an odd function!}}\,dx = \frac{-3}{32}\left[\frac{x^4}{4} - 2x^2\right]_{x=-2}^{x=+2} = \frac{-3}{32}\left[\left(\frac{16}{4}-32\right) - \left(\frac{16}{4}-32\right)\right] = 0$$

c. (5 points) Compute the cdf $F(x)$ for X.

$$\underbrace{F(x) = \Pr(X \le x) = \int_{t=-\infty}^{t=x} f(t)\,dt}_{1\text{ pt.}} = \begin{cases} \int_{t=-2}^{t=x} \frac{-3}{32}(t^2-4)\,dt = \frac{-3}{32}\left[\frac{t^3}{3} - 4t\right]_{t=-2}^{t=x} = \frac{-3}{32}\left[\frac{x^3}{3} - 4x + \frac{16}{3}\right] & \text{if } -2 < x < 2 \quad (2\text{ pts}) \\ \underbrace{0}_{1\text{ pt}} & \text{if } x \le -2 \\ \underbrace{1}_{1\text{ pt}} & \text{if } x \ge +2 \end{cases}$$

d. (5 points) If X_1, X_2 are independent and identically distributed, both with explicit the same distribution as X, then what is $\Pr(X_1 < X_2)$? $\boxed{\text{Write down the integral - do not evaluate it.}}$

Since
$$f(x_1,x_2) = f(x_1)f(x_2) = \begin{cases} \left(\frac{-3}{32}\right)^2 (x_1^2-4)(x_2^2-4) & \text{if } -2<x_1<2 \text{ and } -2<x_2<2 \\ 0 & \text{else} \end{cases}$$

$$\Pr(X_1<X_2) = \int_{x_2=-\infty}^{x_2=+\infty}\int_{x_1=-\infty}^{x_1=x_2} f(x_1,x_2)\,dx_1\,dx_2 = \boxed{\underbrace{\int_{x_2=-2}^{x_2=+2}}_{2\text{ pts}} \underbrace{\int_{x_1=-2}^{x_1=x_2} \left(\frac{-3}{32}\right)^2 (x_1^2-4)(x_2^2-4)\,dx_1\,dx_2}_{3\text{ pts}}} \quad \text{ANSWER:}$$

not required on exam!

$$\frac{1}{2}$$

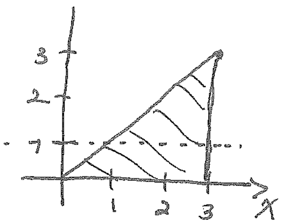
by symmetry, since $\Pr(X_1<X_2) = \Pr(X_1 \ge X_2)$ and $\Pr(X_1<X_2) + \Pr(X_1 \ge X_2) = 1$

$$\begin{array}{r} 32 \\ 32 \\ \hline 64 \\ 96 \\ \hline 1024 \end{array}$$

Problem 5. *(15 points total)* Let X be a continuous random variable, uniformly distributed on the interval $[0, 3]$.

a. (10 points) Let Y be a continuous random variable chosen uniformly on the interval $[0, x]$ after knowing the value $X = x$. Compute the conditional pdf $g_1(x|y = 1) = g_1(x|1)$ for all values of x.

2 pts $f_1(x) \doteq \begin{cases} \frac{1}{3-0} = \frac{1}{3} & \text{for } x \in [0,3] \\ 0 & \text{otherwise} \end{cases}$



2 pts $g_2(y|x) \doteq \text{Unif}[0,x] = \begin{cases} \frac{1}{x-0} = \frac{1}{x} & \text{for } y \in [0,x] \\ 0 & \text{otherwise} \end{cases}$ for $x \in [0,3]$

Hence $f(x,y) = g_2(y|x) f_1(x) = \begin{cases} \frac{1}{3} \cdot \frac{1}{x} = \frac{1}{3x} & \text{for } 0 \le y \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$

$f_2(y) = \int_{x=-\infty}^{x=+\infty} f(x,y)\,dx \Rightarrow f_2(1) = \int_{x=1}^{x=3} \frac{1}{3x}\,dx = \frac{1}{3}\log(x)\Big]_{x=1}^{x=3} = \frac{\log(3) - \log(1)}{3} = \frac{\log(3)}{3}$

2 pts

$g_1(x|y=1) = \frac{f(x,1)}{f_2(1)} = \begin{cases} \frac{1}{3x} \Big/ \frac{\log(3)}{3} = \frac{1}{x\log 3} & \text{if } 1 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$ 2 pts

2 pts

b. (5 points) Compute the pdf $g(z)$ for $Z = X^7$.

Since $Z = X^7$ for $X \in [0,3]$ $\Leftrightarrow$ $X = Z^{\frac{1}{7}} =: s(Z)$ for $Z \in [0, 3^7]$ $\overset{\frac{d}{ds}}{\rightsquigarrow}$ $\overbrace{\frac{1}{7} z^{\frac{-6}{7}}}^{1\text{ pt}}$

2 pts

one has $g(z) = f(s(z)) \left|\frac{ds}{dz}\right|$ where $f(x) \doteq \begin{cases} \frac{1}{3-0} = \frac{1}{3} & \text{for } x \in [0,3] \\ 0 & \text{otherwise} \end{cases}$

$= \begin{cases} \frac{1}{3} \cdot \left|\frac{1}{7} z^{\frac{-6}{7}}\right| = \frac{1}{21} z^{\frac{-6}{7}} & \text{for } z \in [0, 3^7] \\ 0 & \text{otherwise} \end{cases}$

Since $z^{\frac{-6}{7}} = (z^{-\frac{1}{7}})^6 \ge 0$

1 pt

Problem 6. *(15 points total)* A group of n people walk into a restaurant, hand their hat to the hat-check attendant, and after dinner, the attendant hands back one of the hats uniformly at random to each person.

Let X be the random variable which is the number of people that receive their own hat. Compute $\mathbf{E}X$.

(Hint: Try writing X as a sum of simpler *indicator random variables*, that is, random variables that take on values 0 or 1..)

5 pts

Let $X_i = \begin{cases} 1 & \text{if person number } i \text{ gets their own hat back} \\ 0 & \text{otherwise} \end{cases}$

for $i = 1, 2, \ldots, n$

Then $X = X_1 + X_2 + \ldots + X_n$

so $EX = EX_1 + EX_2 + \ldots + EX_n$

where $EX_i = \Pr(\text{person } i \text{ gets their hat back}) \cdot 1 + \Pr(\text{person } i \text{ does not get their hat back}) \cdot 0$

$= \Pr(\text{person } i \text{ gets their hat back}) = \dfrac{(n-1)!}{n!}$ ← ways to assign hats so that person i gets their own / ← ways to randomly assign hats

5 pts

$= \dfrac{1}{n}$

5 pts

Hence $EX = \underbrace{\frac{1}{n} + \frac{1}{n} + \ldots + \frac{1}{n}}_{n \text{ times}}$

$= 1$

7/15 for saying it is $\mathrm{Bin}(n, \frac{1}{n})$