Math 5705 Undergraduate enumerative combinatorics
Spring 2005, Vic Reiner
Midterm exam 1- Due Wednesday February 9, in class

Instructions: This is an open book, open library, open notes, take-home exam, but you are *not* allowed to collaborate. The instructor is the only human source you are allowed to consult. Explain your reasoning- answers without justification or proof (where appropriate) will receive **no credit**.

1. In a standard (American) deck of 52 cards, each card has one of 4 *suits*, (clubs, diamonds, hearts, or spades) and one of 13 *numbers of pips* (2,3,4,5,6,7,8,9,10,J,Q,K,A). Each combination of a suit and number (of pips) occurs exactly once in the deck.

In a game of 5-card poker (with no wild cards), certain hands have standard names.

(a) (10 points) A *flush* (possibly *straight* or *royal*) is a hand containing 5 cards of the same suit. How many different flushes are there?

(b) (10 points) A *full house* is a hand containing 5 cards, in which two cards share the same number (and can be of any suits), while the other three cards share a *different* number (and can be of any suits). How many different full houses are there?

(c) (10 points) *Two pair* is a hand containing 5 cards, in which two cards share the same number (and can be of any suits), while another two cards share a *different* number (and can be of any suits), and the fifth card is of yet a different number (and any suit). How many different two pair hands are there?

2. (20 points) Supplementary problem 9 for Chapter 1 on page 29.

3. Let m, n be two nonnegative integers with $m \geq n$, and let $C_{m,n}$ be the number of lattice paths from $(0,0)$ to (m,n) taking unit steps north or east that stay weakly below the line $y = x$. For example, $C_{n,n}$ is the same as the Catalan number C_n derived in Problem 51.

(a) (15 points) Using ideas from the solution of Problem 51 on page 22, find a formula for $C_{m,n}$.

(b) (5 points) Rewrite your formula for $C_{m,n}$ as an expression that contains no additions or subtractions, i.e. only with multiplications and divisions (e.g. allowing factorials, binomial coefficients, etc.), generalizing the formula

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

4. Prove by any means:

(a) (10 points)

$$\binom{n}{k}\binom{n-k}{m} = \binom{n}{m}\binom{n-m}{k}$$

(This is basically Chapter 1 Supplementary problem 8, but I'm asking for only *one* solution.

(b) (10 points)

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$$

(Hints: if you have no other ideas, here are some you might try. In a lattice path from $(0,0)$ to (n,n), where might you cross the line $y = n - x$? In choosing a committee of n people from a pool of n women and n men, how many women might end up on the committee?)

(c) (10 points)

$$\sum_{m=0}^{n} \binom{m}{k} = \binom{n+1}{k+1}.$$