Math 5705 Undergraduate enumerative combinatorics
Spring 2005, Vic Reiner
Midterm exam 3- Due Wednesday April 13, in class

Instructions: This is an open book, open library, open notes, take-home exam, but you are *not* allowed to collaborate. The instructor is the only human source you are allowed to consult. Explain your reasoning- answers without justification or proof (where appropriate) will receive **no credit**.

1. (30 points) Use the technique of generating functions to solve the following recurrence (which appeared in Problem 211(c)). Your answer should be an explicit formula, not involving any summations, that epresses a_i as a function of a_0 and i:

$$a_i = 3a_{i-1} + 2^i \text{ for } i \geq 1.$$

2. (20 points total) Chapter 4, Supplementary Problem 9, on page 99. There are two questions asked there, each worth 10 points.

3. (30 points total) Let a_k be the number of k-element multisets chosen from a set with n elements in which the multiplicity of each element is less than m (i.e. the multiplicities can be at most $m-1$). Prove the following explicit formula for a_k ...

$$\begin{aligned} a_k = & \sum_{\substack{i\geq 0 \\ j\geq 0 \\ mi+j=k}} (-1)^i \binom{n}{i}\binom{n+j-1}{j} \\ = & \binom{n+k-1}{k} \\ & - \binom{n}{1}\binom{n+k-m-1}{k-m} \\ & + \binom{n}{2}\binom{n+k-2m-1}{k-2m} \\ & - \binom{n}{3}\binom{n+k-3m-1}{k-3m} + \cdots \end{aligned}$$

(a) (15 points) via the technique of generating functions.
(b) (15 points) via the principle of inclusion-exclusion.

4. (20 points total) Recall from Problem 210 on page 90 the definition of the q-binomial coefficient:

$$\begin{bmatrix} N \\ k \end{bmatrix}_q := \sum_{\substack{\text{partitions } \lambda: \\ \lambda_1 \leq N-k \\ \ell(\lambda) \leq k}} q^{|\lambda|} = \sum_{i \geq 0} a_i q^i$$

where a_i is the number of partitions of i with largest part $\lambda_1 \leq N - k$ and number of parts $\ell(\lambda) \leq k$ (that is, λ has Ferrers diagram that fits inside an $k \times N - k$ rectangle).

For example,

$$\begin{bmatrix} 5 \\ 2 \end{bmatrix}_q = 1 + q + 2q^2 + 2q^3 + 2q^4 + q^5 + q^6$$

$$\begin{bmatrix} 7 \\ 3 \end{bmatrix}_q = 1 + q + 2q^2 + 3q^3 + 4q^4 + 4q^5 + 5q^6 + 4q^7 + 4q^8 + 3q^9 + 2q^{10} + q^{11} + q^{12}$$

(a) (10 points) Explain by any means why the sequence of coefficients for $\begin{bmatrix} N \\ k \end{bmatrix}_q$ as a polynomial in q (such as $(1, 1, 2, 2, 2, 1, 1)$ and $(1, 1, 2, 3, 4, 4, 5, 4, 4, 3, 2, 1, 1)$ in the two examples above) is always *palindromic*, that is, the sequence reads the same forwards as backwards.

(b) (10 points) Recall that one version of the binomial theorem asserts

$$(1 + x)^N = \sum_{k \geq 0} \binom{N}{k} x^k.$$

Prove the following *q-binomial theorem*:

$$(1 + x)(1 + qx)(1 + q^2x) \cdots (1 + q^{N-1}x) = \sum_{k \geq 0} q^{\binom{k}{2}} \begin{bmatrix} N \\ k \end{bmatrix}_q x^k.$$