Math 5705 Undergraduate enumerative combinatorics
Fall 2002, Vic Reiner
Midterm exam 2- Due Friday October 11, in class

Instructions: This is an open book, open library, open notes, take-home exam, but you are *not* allowed to collaborate. The instructor is the only human source you are allowed to consult.

1. (15 points) Problem 95 on page 40.

2. (15 points) Supplementary problem 1 for Chapter 2 on pages 49-50. Make sure that you explain your answer.

3. (a) (10 points) How many spanning trees on vertex set $[n] := \{1, 2, \ldots, n\}$ have the vertex labelled 1 as a leaf (=vertex of degree one)?
(b) (10 points) Fix a positive integer k, and let $n \geq k$. What is the probability that the vertices labelled $\{1, 2, \ldots, k\}$ are *all* leaves in a randomly chosen spanning tree on vertex set $[n]$? What value does this probability approach in the limit as n goes to infinity (with k still fixed)?

4. Given a tree T, let $\ell(T)$ denote the number of leaves, and let $m(T)$ denote the maximum of all of the vertex degrees.
(a) (15 points) Prove that $\ell(T) \geq m(T)$.
(b) (5 points) Is it possible to have $\ell(T) > m(T)$? Prove or disprove this.

(Turn over the page for Problem 5)

5. A *partition* of a set S with n *blocks* is a decomposition of S as a disjoint union $S = B_1 \cup \cdots \cup B_n$ where we don't care about the ordering or labelling of the blocks B_i. The number of partitions of a k-element set into n blocks is called the *Stirling number of the second kind* $S(k, n)$, and the total number of partitions of a k-element set into any number of blocks is called the *Bell number* $B(k)$. In other words, $B(k) = \sum_{n=1}^{k} S(k, n)$.

For example, here are the partitions of the 4-element set [4], in which set brackets have been omitted and the blocks are separated by hyphens:

number of blocks:	partitions of [4]
1	1234
2	$1-234$
	$2-134$
	$3-124$
	$4-123$
	$12-34$
	$13-24$
	$14-23$
3	$12-3-4$
	$13-2-4$
	$14-2-3$
	$23-1-4$
	$24-1-3$
	$34-1-2$
4	$1-2-3-4$

which shows that $S(4, 1) = 1, S(4, 2) = 7, S(4, 3) = 6, S(4, 4) = 1$ and $B(4) = 1 + 7 + 6 + 1 = 15$.

(a) (5 points) Find simple explicit formulas as functions of k for

$$S(k, 1),\ S(k, 2),\ S(k, k-1),\ S(k, k).$$

(b) (10 points) This is essentially problem 130 on page 60, so you might want to look at it for hints: Find a recurrence that expresses $S(k, n)$ in terms of $S(k-1, n)$ and $S(k-1, n-1)$, in the same spirit as the Pascal's triangle recurrence for binomial coefficients.

(c) (5 points) Problem 132 on page 60.

(d) (10 points) This is essentially problem 137 on page 61. Find a recurrence that expresses $B(k)$ in terms of $B(0), B(1), \ldots, B(k-1)$ (where as a convention, we decree that $B(0) := 1$).