Math 5705 Undergraduate enumerative combinatorics
Fall 2002, Vic Reiner
Midterm exam 3- Due Friday November 15, in class

Instructions: This is an open book, open library, open notes, take-home exam, but you are *not* allowed to collaborate. The instructor is the only human source you are allowed to consult.

1. (16 points total; 4 points each) Chapter 3, Supplementary problem 2(a),(c),(d),(e) on page 72.

2. (16 points) Chapter 3, Supplementary problem 3 on page 73.

3. (16 points) Chapter 3, Supplementary problem 8 on page 73.

4. (16 points) Chapter 4, Supplementary problem 1 on page 100.

5. (16 points) Chapter 4, Supplementary problem 14 on page 102.

6. This problem is about Stirling numbers of the 1st kind, and is related to #147-149 in the text and also to Chapter 3 Supplementary Problem 10. So feel free to look at those for ideas and hints.

Recall that the *Stirling number of the 2nd kind* $S(k,n)$ is the number of partitions of $[k]$ into n blocks. We showed (Problem # 145) that they are the change-of-basis coefficients in the vector space of polynomials in x of degree at most k, writing the basis $\{1, x, x^2, \dots, x^k\}$ in terms of the basis $\{1, x^{\underline{1}}, x^{\underline{2}} \dots, x^{\underline{k}}\}$:

$$x^k = \sum_{n=0}^{k} S(k,n) x^{\underline{n}},$$

where we further recall that

$$x^{\underline{n}} := x(x-1)(x-2)\cdots(x-n+1)$$
$$x^{\overline{n}} := x(x+1)(x+2)\cdots(x+n-1).$$

Now define the *Stirling number of the 1st kind* $s(k,n)$, and also the *signless Stirling number of the 1st kind* $c(k,n)$ as these change-of-basis coefficients:

$$x^{\underline{k}} = \sum_{n=0}^{k} s(k,n)x^n$$
$$x^{\overline{k}} = \sum_{n=0}^{k} c(k,n)x^n$$

(a)(5 points) Compute $s(4,k)$ and $c(4,k)$ for $k = 1,2,3,4$, and give simple explicit formulas for $s(k,k), c(k,k), s(k,1), c(k,1)$.

(b)(5 points) Explain why $c(k,n)$ is always non-negative, and write down the simple formula relating $s(k,n)$ to $c(k,n)$. (For this reason, when proving facts about $s(k,n)$ or $c(k,n)$, one has a choice about which one to use, and one or the other may be more convenient).

(c)(5 points) This is essentially Problem #147. Find a recurrence that expresses $s(k,n)$ in terms $s(k-1,n)$ and $s(k-1,n-1)$.

(d) (5 points) A *permutation* of $[k]$ is a bijection $\pi : [k] \to [k]$. We will use the following notation:

$$\pi = \begin{pmatrix} 1 & 2 & \cdots & n \\ \pi(1) & \pi(2) & \cdots & \pi(k) \end{pmatrix}.$$

Any permutation decomposes uniquely into *cycles*. For example,

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 2 & 9 & 6 & 4 & 5 & 1 & 8 & 7 \end{pmatrix}$$

decomposes into 4 cycles

$$1 \to 3 \to 9 \to 7 \to 1$$
$$4 \to 6 \to 5 \to 4$$
$$2 \to 2$$
$$8 \to 8$$

Show that $c(k,n)$ is the number of permutations of $[k]$ having exactly n cycles.

(e) (No points; just an "Extra for experts", not required for the exam) Explain why for any m, the two matrices

$$(S(k,n))_{n,k=0,1,\ldots,m}, \qquad (s(k,n))_{n,k=0,1,\ldots,m}$$

are inverse to each other.