Chromatic polynomials (Bondy-Murty §8.4)

In an (unsuccessful) attempt to prove the 4-color Theorem, G.D. Birkhoff (1912) introduced...

DEFINITION: For a multigraph $G=(V,E)$, the chromatic polynomial

$$\underbrace{\pi(G,k)}_{\text{sometimes denoted } \chi(G,k) \text{ or } \chi_k(G) \text{ or } \pi_k(G) \text{ in Bondy-Murty}} := \underbrace{\#\text{ of proper vertex } k\text{-colorings } f: V \to \{1,2,\ldots,k\}}_{\text{not yet clear that it's a polynomial in } k\,!?}$$

sometimes denoted $\chi(G,k)$
or $\chi_k(G)$
or $\pi_k(G)$ in Bondy-Murty

not yet clear that it's a polynomial in k !?

EXAMPLES

① For complete graphs K_n,

$$\pi(K_1, k) = k$$

$$\pi(K_2, k) = k(k-1) = k^2 - k$$

k choices here (1st)   k−1 choices here (2nd)

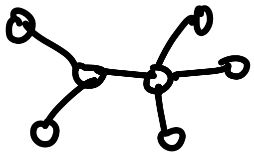
$$\pi(K_3, k) = k(k-1)(k-2) = k^3 - 3k^2 + 2k$$

k   k−1   k−2

and $\pi(K_n, k) = k(k-1)(k-2)\cdots(k-(n-1))$

② ACTIVE LEARNING: Prove this...

PROPOSITION: For any tree $T=(V,E)$, one has $\pi(T,k) = k(k-1)^{|V|-1}$ (part of HW5, Exer. 8.4.3(a))

e.g. $\pi\Big($  $, k\Big) = k(k-1)^6$

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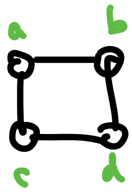
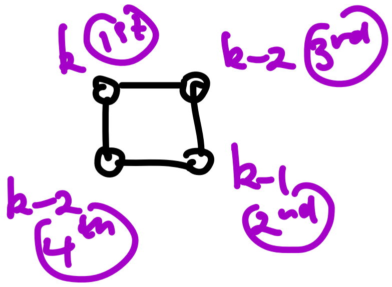
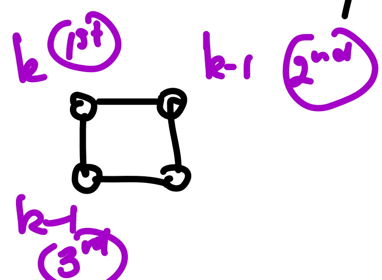
③ $\pi(G,k)$ does not always factor completely as $\pi(G,k) = k(k-r_1)(k-r_2)\cdots(k-r_m)$

e.g. $\pi\Big($  $, k\Big) = \left|\left\{\begin{array}{l}k\text{-colorings}\\ \text{with}\\ f(a)\neq f(d)\end{array}\right\}\right| + \left|\left\{\begin{array}{c}k\text{-colorings}\\ \text{with}\\ f(a)=f(d)\end{array}\right\}\right|$

C_4



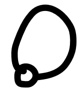


$$= k(k-1)(k-2)^2 + k(k-1)^2$$
$$= k(k-1)\big((k-2)^2 + k-1\big)$$
$$= k(k-1)(k^2-4k+4+k-1)$$
$$= k(k-1)\underbrace{(k^2-3k+3)}_{\text{irreducible over } \mathbb{R}}$$
$$\left[= k^4-4k^3+6k^2-3\right]$$

$\pi(C_4,k)$ doesn't factor completely, but is still a polynomial!

THEOREM (Birkhoff 1912, Birkhoff & Lewis 1946)

Let $G=(V,E)$ be a multigraph.

(i) $\pi(G,k)=0$ if G has any loops 

(ii) For any non-loop edge e of G,

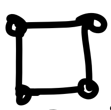
$$\pi(G,k) = \underbrace{\pi(G\setminus e,k)}_{\text{deletion}} - \underbrace{\pi(G/e,k)}_{\text{contraction}}$$

(iii) If G is simple, then $\pi(G,k)$ is a polynomial function of k, of the form

$$\pi(G,k) = k^n - mk^{n-1} + a_{n-2}k^{n-2} - a_{n-3}k^{n-3} + \ldots \pm a_{c(G)}k^{c(G)}$$

where $n:=|V|$, $m=|E|$, $c(G):=$ # of connected components of G with coefficients alternating in sign, so that every $a_i \in \mathbb{Z}_{\geq 1} := \{1,2,3,\ldots\}$.

$n=|V|$ $\qquad$ $m=|E|$ $\qquad$ $c(G)$

EXAMPLE $\pi($  $,k) = k^4 - 4k^3 + 6k^2 - 3k^1$

C_4

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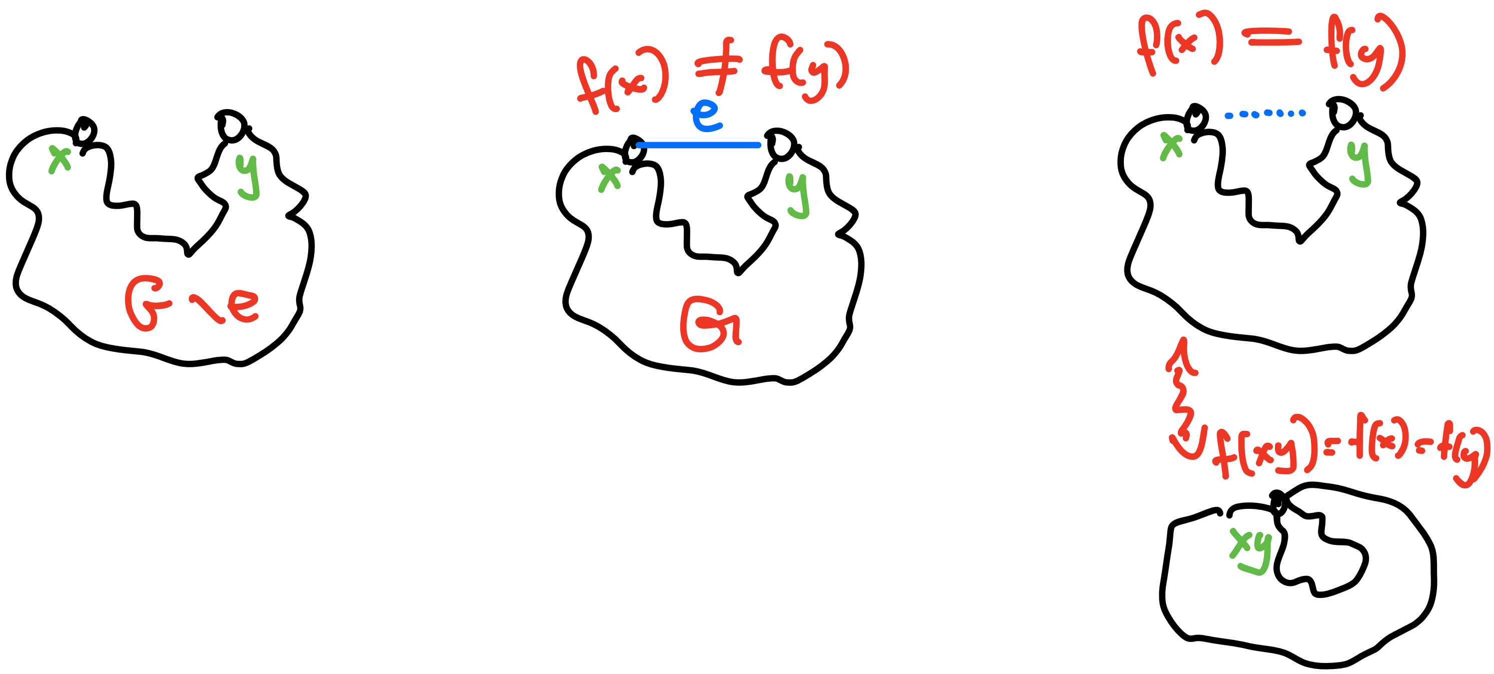
proof: (i) should be clear by convention – if G has a self-loop, it has no proper colorings.

For (ii), let's instead show for a non-loop $e = \{x, y\}$,

$$\pi(G \setminus e, k) = \pi(G, k) + \pi(G/e, k)$$

because...

$$\left\{ \begin{array}{c} \text{proper } k\text{-colorings} \\ \text{of } G \setminus e \end{array} \right\} = \left\{ \begin{array}{c} \text{those with} \\ f(x) \neq f(y) \end{array} \right\} \sqcup \left\{ \begin{array}{c} \text{those with} \\ f(x) = f(y) \end{array} \right\}$$



$$= \left\{ \begin{array}{c} \text{proper } k\text{-colorings} \\ \text{of } G \end{array} \right\} \sqcup \left\{ \begin{array}{c} \text{proper } k\text{-colorings} \\ \text{of } G/e \end{array} \right\}$$

For (iii), show it by induction on $|E|$ using (ii).

BASE CASE: $|E| = 0$. Then $G =$ ⁘ n isolated vertices

$$\pi(G, k) = k^n = k^{c(G)} \checkmark$$

INDUCTIVE STEP: First note that

$$m(G \setminus e) = m - 1 = m(G) - 1$$
$$n(G \setminus e) = n(G) = n$$
$$n(G/e) = n(G) - 1 = n - 1$$
$$c(G/e) = c(G)$$

From (ii),

will vanish if e had parallel copies, which become loops in G/e

$$\pi(G,k) = \underbrace{\pi(G \setminus e, k)} - \pi(G/e,k)$$

$$= k^n - (m-1)k^{n-1} + \hat{a}_{n-2}k^{n-2} - \hat{a}_{n-3}k^{n-3} + \ldots \pm \hat{a}_{c(G)}k^{c(G)}$$

$$- \left(k^{n-1} - \hat{\hat{a}}_{n-2}k^{n-2} + \hat{\hat{a}}_{n-3}k^{n-3} - \ldots \mp \hat{\hat{a}}_{c(G)}k^{c(G)}\right)$$

with $\hat{a}_i, \hat{\hat{a}}_j \in \{1,2,3,\ldots\}$ (except maybe $\hat{\hat{a}}_{c(G)} = 0$)

$$= k^n - m k^{n-1} + a_{n-2}k^{n-2} - a_{n-3}k^{n-3} + \ldots \pm a_{c(G)}k^{c(G)}$$

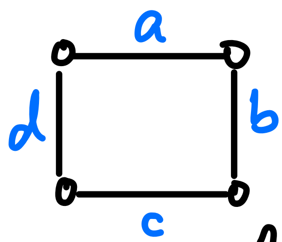
where $a_i = \hat{a}_i + \hat{\hat{a}}_i \in \{1,2,3,\ldots\}$. ■

There is more one can say about the alternating coefficients $a_i \in \{1,2,3,\ldots\}$ appearing in $\pi(G,k)$

DEFINITION: Given $G=(V,E)$, pick an ordering $E = \{e_1 < e_2 < \ldots < e_m\}$ of the edges, and then call a subset $B \subset E$ a broken circuit if there is some cycle $C = \{e_{i_1} < e_{i_2} < \ldots < e_{i_r}\}$ in G and $B = \{ \quad e_{i_2} < \ldots < e_{i_r}\} = C - \underbrace{\{e_{i_1}\}}_{\min C}$

THEOREM (Whitney 1932)

$$\pi(G,k) = \sum_{i=0}^{n} (-1)^i a_{n-i} k^{n-i}$$ where $a_i := \left|\left\{ A \subseteq E : |A| = i \text{ and } A \text{ contains no broken circuits} \right\}\right|$

EXAMPLE $G = C_4 =$ 

$E = \{a < b < c < d\}$

G contains only one cycle $C = \{a, b, c, d\}$
so there is only one broken circuit $B = \{b, c, d\} = C - \{a\}$

A containing no broken circuit	$\lvert A \rvert$		$\pi(C_4, k) =$
$\emptyset$	0	$1 = a_4$	k^4
a	1	$4 = a_3$	$-4k^3$
b	1		
c	1		
d	1		
ab	2	$6 = a_2$	$+6k^2$
ac	2		
ad	2		
bc	2		
bd	2		
cd	2		
abc	3	$3 = a_1$	$-3k^1$
abd	3		
acd	3		

Whitney's Theorem is not so hard to prove by induction on $\lvert E \rvert$ using

$\pi(G, k) = \pi(G \setminus e, k) - \pi(G/e, k)$, but let's skip it.

There is an interesting interpretation for the sum $\sum_i a_i$.

THEOREM: (Stanley's 1973 "(-1)-color Thm.")

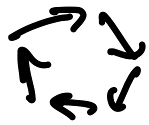
For a multigraph $G=(V,E)$,

$$\sum_{i=1}^{n} a_i = (-1)^n \cdot \pi(G,-1)$$

$$= \#\ \text{acyclic orientations of } E$$

choice of $x \to y$ or $x \leftarrow y$ for each edge $x \overset{e}{—} y$

creating no directed cycles



EXAMPLES

① For trees $T=(V,E)$,

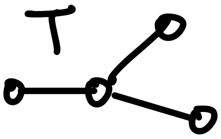
we saw $\pi(T,k) = k(k-1)^{n-1}$

set $k=-1$, multiply by $(-1)^n$

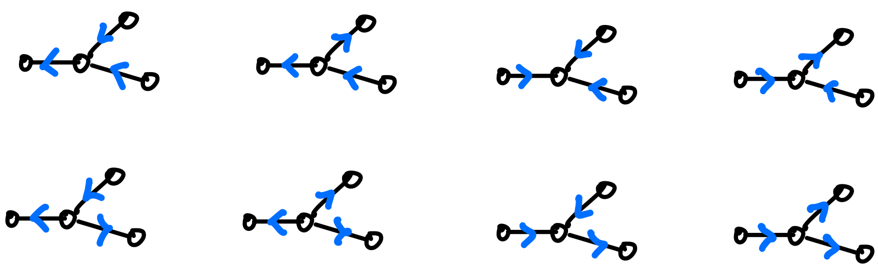
$$(-1)^n \pi(T,-1) = (-1)^n \cdot (-1)(-2)^{n-1}$$

$$= 2^{n-1} = 2^m \quad \text{where } m = |E|$$

T



$= \#$ acyclic orientations of T, since all orientations are acyclic



$\Big\}\ 2^3 = 8$

② For complete graphs,

$$\pi(K_n, k) = k(k-1)(k-2)\cdots(k-(n-1))$$

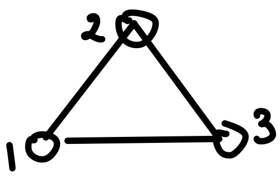
set $k=-1$, multiply by $(-1)^n$

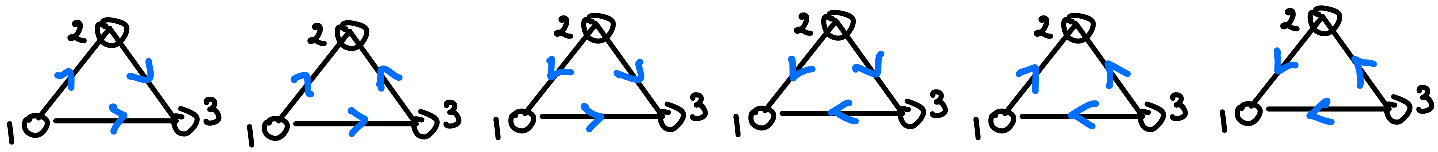
$$(-1)^n \pi(K_n,-1) = (-1)^n(-1)(-2)(-3)\cdots(-n)$$

$$= n!$$

= #acyclic orientations of K_n, since they are in bijection with linear orders on $\{1,2,3,\ldots,n\}$:
make $i \to j$ if $i<j$ in the order

EXAMPLE

K_3 



$1<2<3$ $\quad$ $1<3<2$ $\quad$ $2<1<3$ $\quad$ $2<3<1$ $\quad$ $3<1<2$ $\quad$ $3<2<1$

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proof of Stanley's Thm:

First, since $\pi(G,k) = k^n - a_{n-1}k^{n-1} + a_{n-2}k^{n-2} - \cdots$

$$= \sum_{i=1}^{n} (-1)^{n-i} a_i k^i$$

one has $(-1)^n \pi(G,-1) = (-1)^n \sum_{i=1}^{n} (-1)^{n-i} a_i (-1)^i$

$$= \sum_{i=1}^{n} a_i.$$

Letting $ac(G) := |\{\text{acyclic orientations of } G\}|$,

one can show $ac(G) = (-1)^n \pi(G,-1)$ by induction on $m = |E|$.

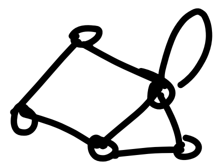
BASE CASE: $m=0$, so $G =$ with no edges

has $\pi(G,k) = k^n$

$$(-1)^n \pi(G,-1) = (-1)^n \cdot (-1)^n = +1 \stackrel{\checkmark}{=} ac(G)$$

INDUCTIVE STEP: $m \geq 1$.

CASE 1: G contains a loop.



Then $\pi(G,k) = 0$

so $$(-1)^n \pi(G,-1) = 0 \stackrel{\checkmark}{=} ac(G)$$

CASE 2: G has no loops.

Pick a non-loop edge $e = \{x,y\}$, and use this...

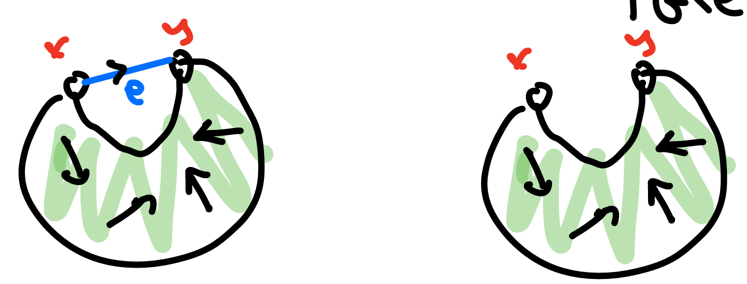
LEMMA: $ac(G) = ac(G \setminus e) + ac(G/e)$

proof: Let $AC(G) := \{\text{acyclic orientations } \Omega \text{ of } G\}$

Then define a map

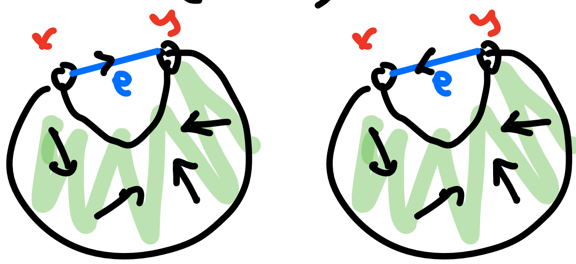
$$\begin{aligned} AC(G) &\xrightarrow{f} AC(G \setminus e) \\ \Omega &\longmapsto \Omega|_{G \setminus e} =: \hat{\Omega} \end{aligned}$$

restrict Ω to $G \setminus e$



We'll count how many pre-images $|f^{-1}(\hat{\Omega})|$ one has for three cases of $\hat{\Omega} \in AC(G \setminus e)$

CASE A: $|f^{-1}(\hat{\Omega})| = 2$



$\Leftrightarrow$ $\hat{\Omega}$ has no directed paths $x \to \dots \to y$ and no directed paths $y \to \dots \to x$

Note that such $\hat{\Omega}$ biject with $AC(G/e)$

CASE B: $|f^{-1}(\hat{\Omega})| = 1$ of form

$\Leftrightarrow$ $\hat{\Omega}$ has a path $x \to \dots \to y$, but none $y \to \dots \to x$

CASE C: $|f^{-1}(\hat{\Omega})| = 1$ of form

$\Leftrightarrow$ $\hat{\Omega}$ has a path $y \to \dots \to x$, but none $x \to \dots \to y$

CASE D: $|f^{-1}(\hat{\Omega})| = 0$ $\Leftrightarrow$ $\hat{\Omega}$ has both paths $x \to \dots \to y$ and $y \to \dots \to x$

IMPOSSIBLE, since $\hat{\Omega}$ acyclic.

Hence $ac(G) = |AC(G)| = \sum_{\hat{\Omega} \in AC(G \setminus e)} |f(\hat{\Omega})|$

$$= 2\underbrace{|\{\hat{\Omega} \text{ in } \underset{\text{CASE A}}{}\}|}_{\alpha :=} + \underbrace{|\hat{\Omega} \text{ in } \underset{\text{CASE B}}{}|}_{\beta :=} + \underbrace{|\hat{\Omega} \text{ in } \underset{\text{CASE C}}{}|}_{\gamma :=}$$

$$= 2\alpha + \beta + \gamma$$

$$= \alpha + (\alpha + \beta + \gamma)$$

$$= ac(G/e) + ac(G \setminus e)$$

proving the LEMMA $\blacksquare$

Now we can induct on $|E|$ to show $ac(G) = (-1)^n \pi(G,-1)$:

$$ac(G) = ac(G \setminus e) + ac(G/e)$$

induction

$$= (-1)^{n(G \setminus e)} \pi(G \setminus e, -1) + (-1)^{n(G/e)} \pi(G/e, -1)$$

$$= (-1)^n \left(\pi(G \setminus e, -1) - \pi(G/e, -1) \right)$$

$\pi(G,k) = \pi(G \setminus e, k) - \pi(G/e, k)$

$$= (-1)^n \pi(G, -1) \quad \blacksquare$$

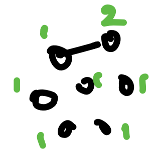
REMARK: The chromatic polynomial

$\pi(G,k)$ has a root at $k=m$

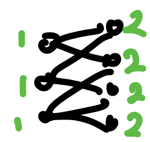
$\iff$ G has no proper vertex m-coloring

$\iff$ $\chi(G) > m$

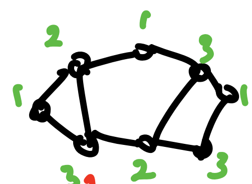
e.g. $\pi(G,1)=0 \iff \chi(G)>1 \iff$ G has an edge



$\pi(G,2)=0 \iff \chi(G)>2 \iff$ G is non-bipartite



On the other hand,



$\pi(G,3)\neq 0 \Leftarrow \chi(G)\leq 3 \Leftarrow$ G outerplanar

$\pi(G,4)\neq 0 \Leftarrow \chi(G)\leq 4 \Leftarrow$ G planar

is a rephrasing of the 4-color Theorem.

Birkhoff hoped to understand roots of chromatic polynomials $\pi(G,k)$ well enough for G planar to show $\pi(G,4)\neq 0$. But nobody has ever completed this approach, so far.