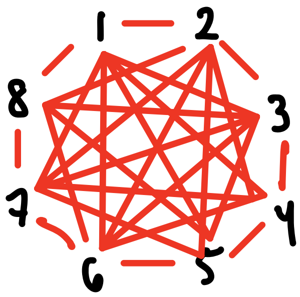
Ramsey Theory (Bondy-Murty § 7.2) and The Probabilistic Method (Alon & Spencer)

QUESTION: If there are 8 people in a room, must there always be

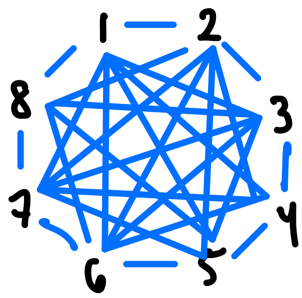
either 3 mutual acquaintances (red K_3 △ below)

or 3 mutual strangers (blue K_3 △ below)

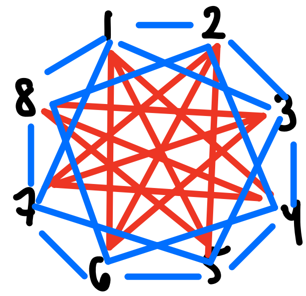
among them?



all acquainted



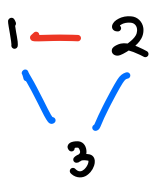
all strangers

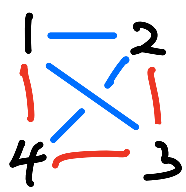


no 3 mutual acquaintances, but, e.g., {1,2,3} all strangers

When only 3 or 4 people are in the room, a (red or blue) monochromatic K_3 can be easily avoided,

e.g.

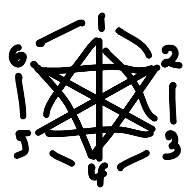




ACTIVE LEARNING:

(a) Can you avoid it for $n=5$?

(b) Can you avoid it for $n=6$?



DEFINITION: For $s,t \in \{2,3,4,\dots\}$, the (2-color) Ramsey number $R(s,t)$ is the smallest n such that any edge 2-coloring (red/blue) of K_n either contains a red K_s or a blue K_t.

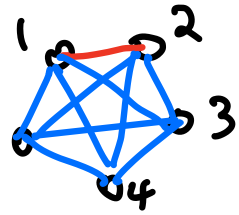
EXAMPLE: $R(3,3)=6$ from above ACTIVE LEARNING

We will soon see that such $R(s,t)$ exist, and get some upper/lower bounds for them.

EXAMPLES:

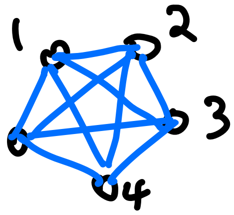
① $R(2,t) = ?$

$R(2,t) \le t$ since either some edge is red



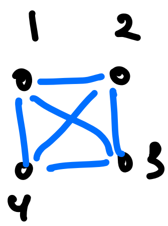
t=5

or all edges are blue



t=5

$R(2,t) > t-1$ since one can color K_{t-1} all blue



Hence $R(2,t)=t$.

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② $R(s,t) = R(t,s)$ by swapping roles of red/blue

③ **PROPOSITION:** (Ramsey 1930) For every $s,t \geq 2$,

(a) $R(s,t)$ exists,

(b) in fact, $R(s,t) \leq R(s,t-1) + R(s-1,t)$ for $s,t \geq 3$

(c) implying $R(s,t) \leq \binom{s+t-2}{s-1} \left(= \binom{s+t-2}{t-1} \right)$

proof: Let's prove all three by induction on $s+t$.

BASE CASE: $s=2$ (or $t=2$ by symmetry).

We saw $R(2,t)=t$ ✓

and $\binom{s+t-2}{s-1} = \binom{t}{1} = t$

INDUCTIVE STEP: $s,t \geq 3$

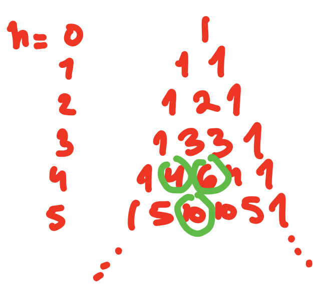
So assume $R(s-1,t)$ exists $\left(\text{and is} \leq \binom{s+t-3}{s-2}\right)$

$R(s,t-1)$ exists $\left(\text{and is} \leq \binom{s+t-3}{s-1}\right)$

Let $n := R(s-1,t) + R(s,t-1)$

$$\left(\leq \binom{s+t-3}{s-2} + \binom{s+t-3}{s-1} = \binom{s+t-2}{s-1} \right)$$

(Pascal's triangle recurrence $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$)

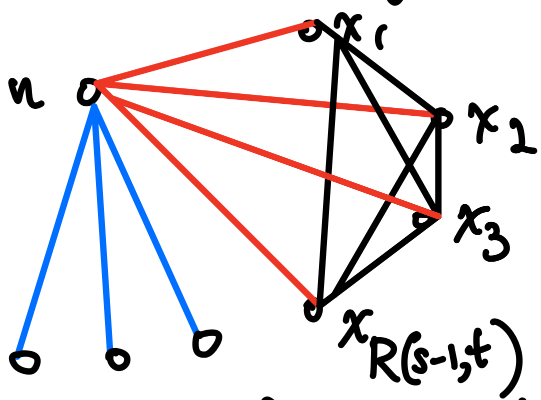


We'll show $R(s,t) \le n$ by showing that any red/blue edge coloring of K_n contains either a red K_s or a blue K_t.

Consider vertex n, which has $n-1$ neighbors, and hence either touches $\geq R(s-1,t)$ red edges or touches $\geq R(s,t-1)$ blue edges

CASE 1: Vertex n touches $\geq R(s-1,t)$ red edges.

Label some of its red edge neighbors $x_1, x_2, \ldots, x_{R(s-1,t)}$



and note that they form a $K_{R(s-1,t)}$, edge 2-colored, so by induction, it must contain either

- a red K_{s-1}, which forms a red K_s together with n
- or it already contains a blue K_t.

CASE 2: Vertex n touches $\geq R(s,t-1)$ blue edges.

Similarly label some blue neighbors $x_1, x_2, \ldots, x_{R(s,t-1)}$, whose $K_{R(s,t-1)}$ either contains a red K_s, or a blue K_{t-1}, making a blue K_t with vertex n. ▨

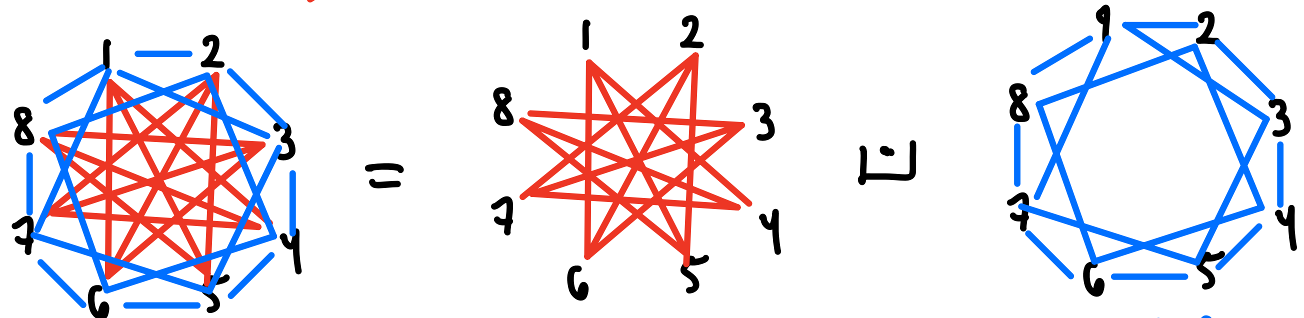
Not very many $R(s,t)$ with $s,t \geq 3$ are known exactly:

$t =$	3	4	5	6	7	8	9	10
$s = 3$	6 $= \binom{4}{2}$	9 (see below)	14	18	23	28	36	40-41
4		18	25	36-40	49-58	59-79	73-105	92-135
5			43-46	59-85	80-133	...	...	...
			⋮	⋮	⋮	⋱		

(See the Wikipedia page on Ramsey's Theorem.)

PROPOSITION: $R(3,4) = 9$

proof: $R(3,4) > 8$ via this coloring of K_8:



no K_3's   no K_4's

To show $R(3,4) \leq 9$, note that in any red-blue edge-coloring of K_9, every vertex $x \in V$ has degree $9-1=8$ so it either

- touches at least $4 = R(2,4)$ red edges ⇝ done as before.
- or touches at least $6 = R(3,3)$ blue edges ⇝ done as before.

Or else every $x \in V$ touches 3 red + 5 blue vertices, which is impossible, since then the red subgraph $H \subset K_9$ has

$$\underbrace{2|E(H)|}_{\text{even}} = \sum_{x \in V} \deg_H(x) = 9 \cdot 3 = \underbrace{27}_{\text{odd}}. \quad \blacksquare$$

Alleged Paul Erdös (1913 – 1996) paraphrased quote:

"If aliens demanded humans to calculate $R(5,5)$ or face attack, we would probably mobilize the resources to do it. If they asked for $R(6,6)$, we should consider attacking them first."

People have instead focussed on asymptotic upper and lower bounds, e.g., ...

COROLLARY: $R(s,t) \leq \binom{s+t-2}{s-1}$

$$\Rightarrow R(s,s) \leq \binom{2(s-1)}{s-1} \approx \frac{4^{s-1}}{\sqrt{s}}$$

Diagonal Ramsey number

as $s \to \infty$

proof: Apply Stirling's approximation

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \quad \text{as } n \to \infty$$

to

$$\binom{2(s-1)}{s-1} = \frac{[2(s-1)]!}{(s-1)!\,(s-1)!} \approx \frac{\sqrt{2\pi \cdot 2(s-1)}\left(\frac{2(s-1)}{e}\right)^{2(s-1)}}{\left[\sqrt{2\pi(s-1)}\left(\frac{s-1}{e}\right)^{s-1}\right]^2}$$

$$= \frac{\sqrt{4\pi(s-1)}}{2\pi(s-1)} \frac{[2(s-1)]^{2(s-1)}}{(s-1)^{2(s-1)}}$$

$$= \frac{2^{2(s-1)}}{\sqrt{\pi(s-1)}} \approx \frac{4^{s-1}}{\sqrt{s}} \quad ▨$$

What about lower bounds on $R(s,s)$?

For that Erdös (1947) introduced ...

The Probabilistic Method

(Ref: "The Probabilistic Method" by Alon & Spencer – on syllabus page)

After quickly reviewing some easy probability, we'll show $R(s,s) > \frac{s}{e\sqrt{2}}(\sqrt{2})^s$ for s large, $(s \gg 0)$

by showing that for $s \gg 0$ if $n > \frac{s}{e\sqrt{2}}(\sqrt{2})^s$, then a random 2-coloring of the edges of K_n has its expected number of monochromatic K_3's (red or blue) less than 1, so one of these 2-colorings with zero monochromatic triangles exists (!)

DEFINITION: A finite probability space Ω is a set $\Omega = \{\omega_1, \omega_2, \dots, \omega_N\}$ with a function

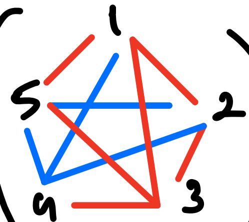
$$\mathbb{P}: \Omega \to \mathbb{R}_{\geq 0}, \quad \omega \mapsto \mathbb{P}(\omega)$$

such that $\sum_{\omega \in \Omega} \mathbb{P}(\omega) = 1$.

EXAMPLES:

① The uniform distribution/probability space on Ω makes all $\mathbb{P}(\omega)$ equal, so $\mathbb{P}(\omega) = \frac{1}{|\Omega|}$.

In particular, if $\Omega = \{\text{all edge 2-colorings of } K_n\}$ then $\mathbb{P}(\omega) = \frac{1}{2^{\binom{n}{2}}}$ for each coloring ω

$n=5$: $\mathbb{P}\Big($  $\Big) = \frac{1}{2^{\binom{5}{2}}} = \frac{1}{2^{10}} = \frac{1}{1024}$

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② There is a probability distribution on

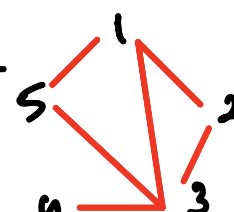
$$\Omega = \{\text{all edge-subgraphs } G = (V, E) \subseteq K_n\}, \quad V = \{x_1, \ldots, x_n\}$$

called $G(n,p) :=$ the Erdös-Rényi random graph with edge probability p

where one includes or excludes each edge $\{i,j\}$, $1 \leq i < j \leq n$, after flipping an unfair coin having

$\mathbb{P}(\text{heads}) = p$ (include $\{i,j\}$ in G)

$\mathbb{P}(\text{tails}) = 1-p$ (exclude $\{i,j\}$ in G)

$\Rightarrow \mathbb{P}(G) = \frac{1}{p^{|E(G)|} \cdot (1-p)^{|E(\bar{G})|}}$, e.g. $\mathbb{P}\Big($  $\Big) = \frac{1}{p^6 \cdot (1-p)^4}$

DEFINITION: An event is any subset $A \subseteq \Omega$, and its probability is

$$\mathbb{P}(A) := \sum_{\omega \in A} \mathbb{P}(\omega)$$

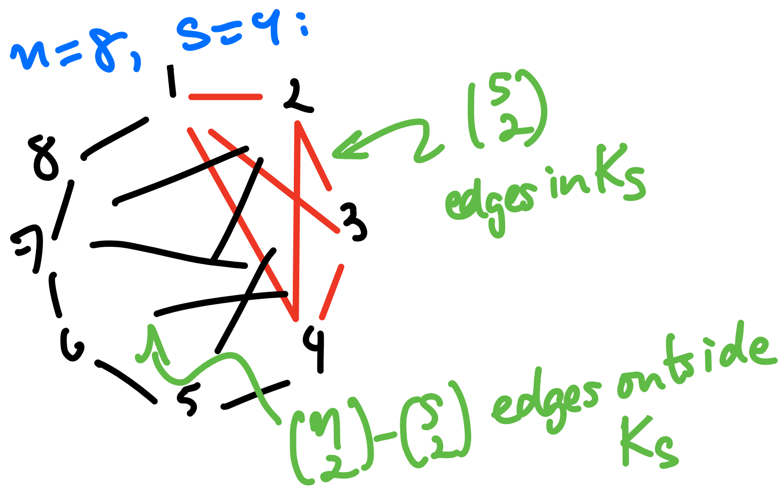
$$= \frac{|A|}{|\Omega|} \text{ if } \Omega \text{ has uniform distribution}$$

EXAMPLE: For $\Omega = \{\text{edge 2-colorings of } K_n\}$ with uniform distribution, the event

$A = \{\text{vertices } 1, 2, \ldots, s \text{ form a red } K_s \text{ inside } K_n\}$

has

$$\mathbb{P}(A) = \frac{|A|}{|\Omega|} = \frac{2^{\binom{n}{2} - \binom{s}{2}}}{2^{\binom{n}{2}}} = 2^{-\binom{s}{2}} = \left(\frac{1}{2}\right)^{\binom{s}{2}}$$



DEFINITION: A random variable is a function $X : \Omega \to \mathbb{R}$

and its expectation (= expected value = mean)

$$\mathbb{E}X := \sum_{\omega \in \Omega} \mathbb{P}(\omega) \cdot X(\omega)$$

EXAMPLES:

① For any event $A \subseteq \Omega$, the indicator random variable $\mathbb{1}_A : \Omega \to \{0,1\}$

sends $\omega \mapsto \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A. \end{cases}$

Hence it has $\mathbb{E}(\mathbb{1}_A) = \sum_{\omega \in \Omega} \mathbb{P}(\omega) \cdot \mathbb{1}_A(\omega)$

$= \sum_{\omega \in A} \mathbb{P}(\omega) = \mathbb{P}(A)$

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② For $\Omega = \{\text{edge 2-colorings of } K_n\}$

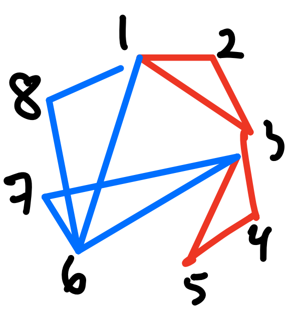
the random variable $X_n : \Omega \longrightarrow \{0,1,2,\ldots\}$

$\omega \longmapsto$ # of monochromatic K_s's inside K_n

can be written as sum of indicator random variables

$$X_n = \mathbb{1}_{A_1} + \mathbb{1}_{A_2} + \ldots + \mathbb{1}_{A_{\binom{n}{s}}} + \mathbb{1}_{B_1} + \mathbb{1}_{B_2} + \ldots + \mathbb{1}_{B_{\binom{n}{s}}}$$

where $A_i := \{\omega \text{ having all red } K_s \text{ on the } i^{th} \text{ s-subset } \{x_1,\ldots,x_s\} \subset V\}$



$B_i := \{\omega \text{ having all blue } K_s \text{ on the } i^{th} \text{ s-subset } \{x_1,\ldots,x_s\} \subset V\}$

An easy, but important fact:

PROPOSITION: (Linearity of Expectation) If $X: \Omega \to \mathbb{R}$ has

$$X = c_1 X_1 + \dots + c_N X_N$$

for some $c_i \in \mathbb{R}$ and $X_i : \Omega \to \mathbb{R}$

then $\mathbb{E}X = c_1 \mathbb{E}X_1 + \dots + c_N \mathbb{E}X_N$.

proof:

$$\mathbb{E}X = \sum_{\omega \in \Omega} \mathbb{P}(\omega) X(\omega)$$

$$= \sum_{\omega \in \Omega} \mathbb{P}(\omega) \sum_{i=1}^{N} c_i X_i(\omega)$$

$$= \sum_{i=1}^{N} c_i \sum_{\omega \in \Omega} \mathbb{P}(\omega) X_i(\omega) = \sum_{i=1}^{N} c_i \mathbb{E}X_i \qquad \blacksquare$$

COROLLARY: For $\Omega = \{\text{edge 2-colorings of } K_n\}$ with uniform distribution, $X_n = \#$ of monochromatic K_s's

has $\mathbb{E}X_n = \binom{n}{2} 2^{1-\binom{s}{2}}$

proof:

$$X_n = \sum_{i=1}^{\binom{n}{s}} \mathbb{1}_{A_i} + \sum_{i=1}^{\binom{n}{s}} \mathbb{1}_{B_i}$$

$$\Rightarrow \mathbb{E}X_n = \sum_{i=1}^{\binom{n}{s}} \left(\mathbb{E}(\mathbb{1}_{A_i}) + \mathbb{E}(\mathbb{1}_{B_i})\right) = \binom{n}{s}\left(2^{-\binom{s}{2}} + 2^{-\binom{s}{2}}\right)$$

$$= \binom{n}{s} 2^{1-\binom{s}{2}} \qquad \blacksquare$$

THEOREM (Erdős 1947) For $s>>0$, whenever $n< \frac{s}{e\sqrt{2}}(\sqrt{2})^s$ then $X_n = \#$ monochromatic K_s's in K_n has $\mathbb{E}X_n < 1$, and hence some edge 2-coloring ω of K_n with no monochromatic K_s's exist.

In particular, $R(s,s) > \frac{s}{e\sqrt{2}}(\sqrt{2})^s$.

Hence $\frac{s}{e\sqrt{2}}(\sqrt{2})^s < R(s,s) < \frac{4^{s-1}}{\sqrt{s}}$

take $\sqrt[s]{(-)} = (-)^{1/s}$, then $\lim_{s\to\infty}(-)$

$$\sqrt{2} < \lim_{s\to\infty} R(s,s)^{\frac{1}{s}} < 4$$

REMARK: Nobody knows which of $\sqrt{2}$, 4 is closer to the truth, or if the limit exists.

Proof of THEOREM:

Assuming $n \le \frac{s}{e\sqrt{2}}(\sqrt{2})^s = \frac{s}{e}2^{\frac{s-1}{2}}$, one has

$$\mathbb{E}X_n = \binom{n}{s}2^{1-\binom{s}{2}} = \frac{\overbrace{n}^{\le n}\overbrace{(n-1)}^{\le n}\overbrace{(n-2)}^{\le n}\cdots\overbrace{(n-(s-1))}^{\le n}}{s!}\cdot 2^{1-\binom{s}{2}}$$

$$\le \frac{n^s\cdot 2}{s!\,2^{\binom{s}{2}}}$$

since we assumed $n \le \frac{s}{e} \cdot 2^{\frac{s-1}{2}}$

$$\le \frac{\left[\left(\frac{s}{e}\right) 2^{\frac{s-1}{2}}\right]^s \cdot 2}{s!\, 2^{\binom{s}{2}}} = \frac{\left(\frac{s}{e}\right)^s \cdot 2^{\frac{s(s-1)}{2}} \cdot 2}{s!\, 2^{\frac{s(s-1)}{2}}} = \frac{\left(\frac{s}{e}\right)^s \cdot 2}{s!}$$

Stirling's approximation: $s! \approx \sqrt{2\pi s} \left(\frac{s}{e}\right)^s$

$$\approx \frac{\left(\frac{s}{e}\right)^s \cdot 2}{\sqrt{2\pi s} \left(\frac{s}{e}\right)^s} = \frac{2}{\sqrt{2\pi s}} < 1 \text{ since } s \gg 0. \blacksquare$$

REMARK: We saw $R(2,t) = t$.

For $R(3,t)$, the asymptotics are known better than the situation for $R(s,s)$:

$$c \cdot \frac{t^2}{\log t} \le R(3,t) \le c' \cdot \frac{t^2}{\log t}$$

for some constants c, c'.

Lower bounds were again proven first by Erdös probabilistically.

Another of Erdös's early applications of the Probabilistic Method ...

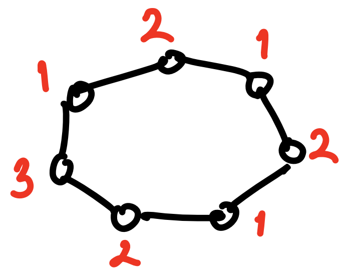
High girth and chromatic number (Bondy-Murty §8.5)

We've noted $\chi(G) \geq \omega(G)$,

chromatic number — max clique size

and the inequality can be strict,

e.g. $\chi(C_m) = 3 > 2 = \omega(C_m)$
m odd



One might wonder whether one can bound $\chi(G)$ in terms of $\omega(G)$, but this construction shows that can't happen:

THEOREM (Mycielski 1955) There exist triangle-free graphs $G_2, G_3, G_4, \ldots$ with $\chi(G_n) = n$.
(so $\omega(G_n) = 2$)

proof: Let $G_2 = K_2 =$ x_1 —— x_2,

and if G_n has vertices $x_1, x_2, \ldots, x_m$,

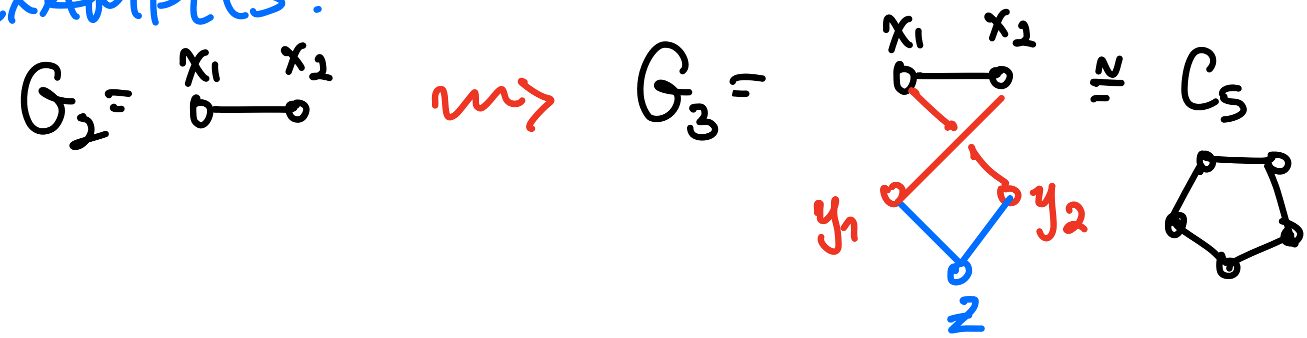
then construct G_{n+1} on $V = \{x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_m, z\}$

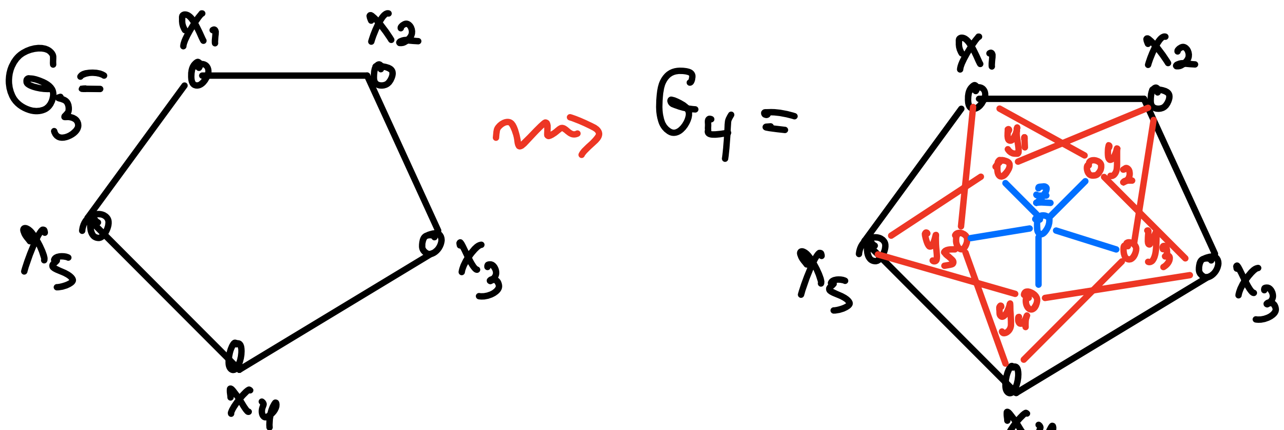
with edges $\{x_i, x_j\}$ from G_n

$\{y_i, z\}$ for $i = 1, 2, \ldots, m$

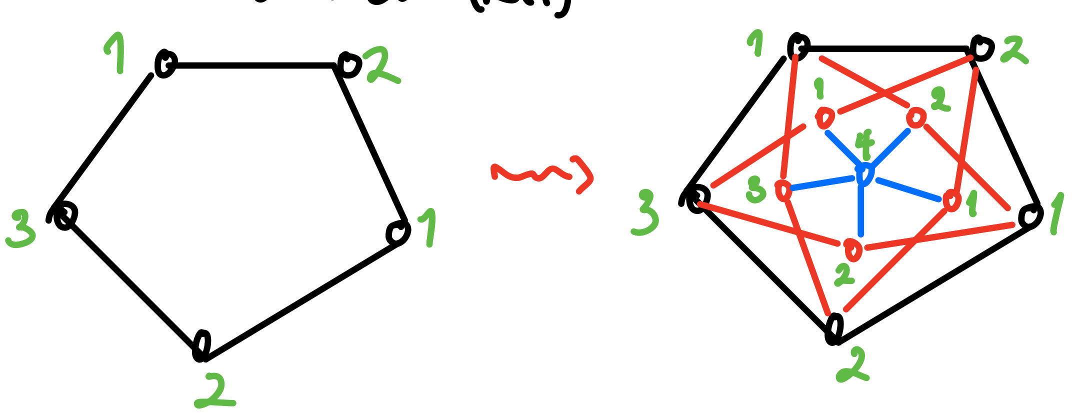
$\{y_i, x_j\}$ for each $i = 1, 2, \ldots, m$ and neighbors x_j of x_i in G

EXAMPLES:





It's not hard to show $\chi(G_{n+1}) \leq 1 + \chi(G_n)$, since a proper k-coloring of G_n leads to one for G_{n+1} in which x_i, y_i get the same color as x_i before, and z gets a new $(k+1)^{st}$ color:



On the other hand, one can show $\chi(G_{n+1}) \geq 1 + \chi(G_n)$, since a proper k-coloring f of G_{n+1} leads to a proper $(k-1)$-coloring $\hat{f}$ of G_n in which $\hat{f}(x_i) := f(y_i)$. This only uses $k-1$ colors since it avoids using $f(z)\ (\neq f(y_i)\ \forall i)$. ▨

This Mycielski construction produces graphs G_n with $\chi(G_n) = n$, but $\text{girth}(G_n) = 4$ for $n \geq 4$.

girth = length of shortest cycle

So one might still hope to upper bound $\chi(G)$ when $\text{girth}(G)$ is high. Also hopeless, due to...

THEOREM: (Erdös 1959) For every k, ℓ, $\exists$ graphs G with $\text{girth}(G) \geq k$, and $\chi(G) \geq \ell$.

His proof is again a probabilistic argument, and uses one more very common and easy-to-prove estimate:

PROPOSITION: ("Markov's inequality" 1884, Chebyshev 1867)

A nonnegative random variable $X: \Omega \to \mathbb{R}_{\geq 0}$ (so $X(\omega) \geq 0 \; \forall \omega \in \Omega$) has for any $t \geq 0$ this tail estimate:

$$\mathbb{P}(X \geq t) \leq \frac{\mathbb{E}X}{t}$$

proof:

$$\mathbb{E}X = \sum_{\omega \in \Omega} \mathbb{P}(\omega) \cdot X(\omega)$$

$$= \sum_{\substack{\omega \in \Omega: \\ X(\omega) \geq t}} \mathbb{P}(\omega) \cdot \underbrace{X(\omega)}_{\geq t} + \underbrace{\sum_{\substack{\omega \in \Omega: \\ |X(\omega)|}} \underbrace{\mathbb{P}(\omega)}_{\geq 0} \cdot \underbrace{X(\omega)}_{\geq 0}}_{\geq 0}$$

$$\geq t \sum_{\substack{\omega \in \Omega: \\ |X| \geq t}} \mathbb{P}(\omega) = t \cdot \mathbb{P}(X \geq t)$$

$$\Rightarrow \frac{\mathbb{E}X}{t} \geq \mathbb{P}(X \geq t) \quad \blacksquare$$

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proof of Erdös's Theorem:

Given k, l, we'll find G with $\chi(G) \geq k$, $\text{girth}(G) \geq l$ by picking it randomly from

$\Omega = G(n,p) =$ Erdös-Rényi random (simple) graph on $\{1, 2, \ldots, n\}$ with edge probability p

so $\mathbb{P}(G) = p^{|E(G)|}(1-p)^{|E(\bar{G})|}$.

We'll consider two random variables on Ω:

- $X : \Omega \to \{0,1,2,\dots\}$
  $G \mapsto X(G) := \{\#\text{ of cycles in } G \text{ of length} < \ell\}$

- $\alpha : \Omega \to \{1,2,3,\dots\}$
  $G \mapsto \alpha(G) :=$ max size of an indep. set of vertices V'.

The (clever) idea is to pick later both

- the edge probability $p = p(n)$ as a function of n
  AND
- an independent size lower bound function $f(n)$

in such a way that three things ①, ②, ③ occur:

① $\mathbb{P}\left(X(G) \geq \frac{n}{2} \right) < \frac{1}{2}$ for $n \gg 0$

② $\mathbb{P}\left(\alpha(G) \geq f(n) \right) < \frac{1}{2}$ for $n \gg 0$

(max indep. set size)

③ $n/f(n) \longrightarrow \infty$ for $n \gg 0$

Then $\mathbb{P}\left(X(G) \geq \frac{n}{2} \text{ OR } \alpha(G) \geq f(n) \right) < \frac{1}{2} + \frac{1}{2} = 1$

and hence $\mathbb{P}\left(X(G) < \frac{n}{2} \text{ AND } \alpha(G) < f(n) \right) > 0$

so there is at least one such G.

Removing one vertex from each cycle of size $< \ell$ in G yields a graph H with $\text{girth}(H) \geq \ell$

$$|V(H)| \geq n - X(G) \geq n - \frac{n}{2} = \frac{n}{2}$$

$$\alpha(H) \leq \alpha(G) < f(n)$$

This leads to a lower bound on $\chi(H)$ using the inequality $\chi(H)\,\alpha(H) \geq |V(H)|$ valid for all graphs

[proof: A proper $\chi(H)$-coloring gives

$V(H) = V_1 \sqcup V_2 \sqcup \dots \sqcup V_{\chi(H)}$ with each V_i indep.

$$\Rightarrow |V(H)| = \sum_{i=1}^{\chi(G)} \underbrace{|V_i|}_{\leq \alpha(H)} \leq \chi(H)\,\alpha(H)$$]

So $$\chi(H) \geq \frac{|V(H)|}{\alpha(H)} \geq \frac{n/2}{f(n)} \xrightarrow{\text{by } ③} \infty$$

$> k$ for any choice of k.

Now we show how to pick $p = p(n)$ and $f(n)$ to get ①, ②, ③.

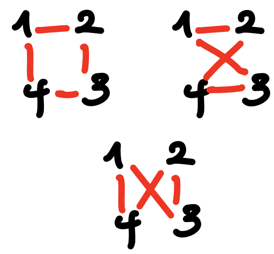
For ①, choose $p = n^{\theta - 1}$ where $0 < \theta < \frac{1}{\ell}$

and let's bound $\mathbb{E}X$ for

$X(G) := \#\{\text{cycles of } G \text{ of length} < \ell\}$.

Note $X = \sum_{\substack{\text{subsets} \\ C \subset \{1,2,\ldots,n\} \\ \text{with } |C| < \ell}} X_C$ where $X_C = \mathbb{1}_{\{G \text{ has a cycle on vertex set } C\}}$

$|C| = 4$: $\frac{(|C|-1)!}{2} = 3$



$$\Rightarrow \mathbb{E}X = \sum_{\substack{C \subset \{1,2,\ldots,n\}: \\ |C| < \ell}} \mathbb{E}X_C$$

$$= \sum_{\substack{C \subset \{1,2,\ldots,n\}: \\ |C| < \ell}} \underbrace{\mathbb{P}(G \text{ has a cycle on } C)}_{\leq p^{|C|} \cdot \frac{(|C|-1)!}{2}}$$

$$\leq \sum_{j=3}^{\ell-1} \binom{n}{j} p^j \frac{(j-1)!}{2}$$

$$= \sum_{j=3}^{\ell-1} \frac{\underbrace{n}_{\leq n}\underbrace{(n-1)}_{\leq n}\underbrace{(n-2)}_{\leq n} \cdots \underbrace{(n-(j-1))}_{\leq n}}{j!} p^j \frac{(j-1)!}{2}$$

$$\leq \sum_{j=3}^{\ell-1} \frac{n^j p^j}{2j} = \sum_{j=3}^{\ell-1} \frac{n^j (n^{\theta-1})^j}{2j} = \sum_{j=3}^{\ell-1} \frac{n^{\theta j}}{2j}$$

Markov's Inequality

$$\Longrightarrow \mathbb{P}\left(X \geq \frac{n}{2}\right) \leq \frac{\mathbb{E}X}{n/2} \leq \frac{2}{n} \sum_{j=3}^{\ell-1} \frac{n^{\theta j}}{2j}$$

$$= \underbrace{\frac{n^{3\theta-1}}{3}}_{} + \underbrace{\frac{n^{4\theta-1}}{4}}_{} + \ldots + \underbrace{\frac{n^{(\ell-1)\theta-1}}{\ell-1}}_{}$$

approach 0 as $n \to \infty$ since $0 < \theta < \frac{1}{\ell}$

$\longrightarrow \infty$ as $n \to \infty$, so $\mathbb{P}(X \geq \frac{n}{2}) < \frac{1}{2}$. (1)

For ②, once we have chosen $f = f(n)$, we'll have

$$\mathbb{P}(\alpha(G) \geq f) = \mathbb{P}\Big(\bigcup_{\substack{S \subset \{1,2,\dots,n\}: \\ |S| = f}} \{S \text{ is stable in } G\} \Big)$$

$$\leq \sum_{\substack{S \subset \{1,2,\dots,n\}: \\ |S| = f}} \mathbb{P}(S \text{ is stable in } G)$$

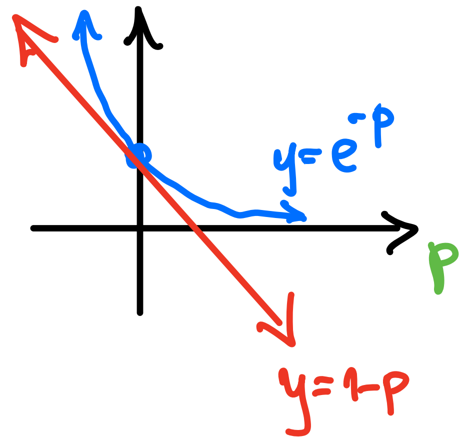
$$= \binom{n}{f} (1-p)^{\binom{f}{2}}$$

$$\leq \frac{n^f}{f!} \cdot (e^{-p})^{\binom{f}{2}}$$

$$< n^f (e^{-p})^{\frac{f(f-1)}{2}}$$

$$= \left(n e^{-\frac{p(f-1)}{2}} \right)^f$$

$e^{-p} \geq 1-p$ for $p > 0$ from



Erdös suggests choosing $f = f(n) = \frac{3}{p} \log n = 3n^{1-\theta} \cdot \log n$

so that $$\mathbb{P}(\alpha(G) \geq f(n)) < \left(n e^{-\frac{p}{2}\left(\frac{3}{p}\log n - 1\right)} \right)^{f(n)}$$

$$= \left(n e^{-\frac{3}{2}\log n + \frac{p}{2}} \right)^{f(n)}$$

for $n \to \infty$, since $0 < p < 1$

$$\approx \left(n e^{-\frac{3}{2}\log n} \right)^{f(n)} = \left(n \cdot n^{-\frac{3}{2}} \right)^{f(n)}$$

$$= n^{-\frac{1}{2} f(n)} = \frac{1}{(\sqrt{n})^{f(n)}} \longrightarrow 0 \text{ as } n \to \infty$$

so $\mathbb{P}(\alpha(G) \geq f(n)) \overset{②}{<} \frac{1}{2}$ for $n >> 0$.

Lastly, note these choices also make ③ happen:

$$\frac{n}{f(n)} = \frac{n}{3n^{1-\theta}\cdot\log n} = \frac{1}{3}\cdot\frac{n^{\theta}}{\log n}$$

$$\longrightarrow \infty \quad \text{as } n\to\infty$$

since $\theta > 0$. ▨

REMARK:

Erdös & Rényi 1960 initiated the study of the *expected* structure of a graph G in $G(n,p)$

as p grows in the range $0<p<1$, as $n\to\infty$

e.g. when does it start to have

large connected components?

(A: around $p=\frac{1}{2}$)

How large are those components?

girth(G)?

Max clique size $\omega(G)$?

indep set size $\alpha(G)$?

See Part II of the book by Alon & Spencer for lots on these topics.