Exploring planar duality (Bondy-Murty § 9.2)

Recall from our discussion of planar maps...

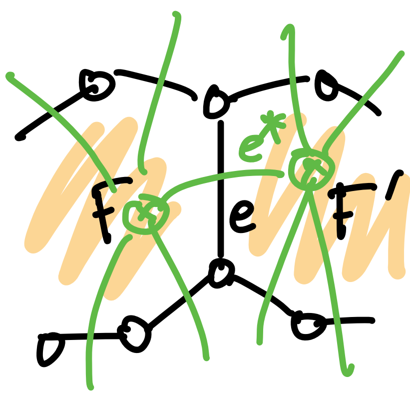
DEFINITION: For a plane graph $G=(V,E)$,

i.e. a planar multigraph, together with a choice of embedding into $\mathbb{R}^2$ without crossings

one defines its plane dual $G^* = (V^*, E^*)$

$V^* = \{\text{faces } F \text{ of } G\}$

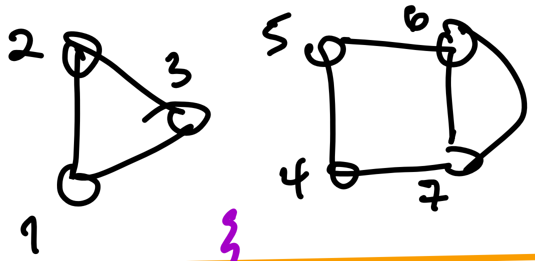
$E^* = \{e^* = \{F,F'\} : e \in E\}$ where e bounds both faces F, F'



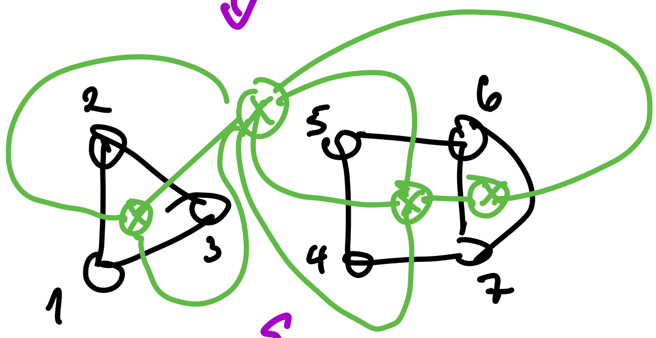
There are a few subtleties about $G \rightsquigarrow G^*$, and lots of cool properties.

EXAMPLES:

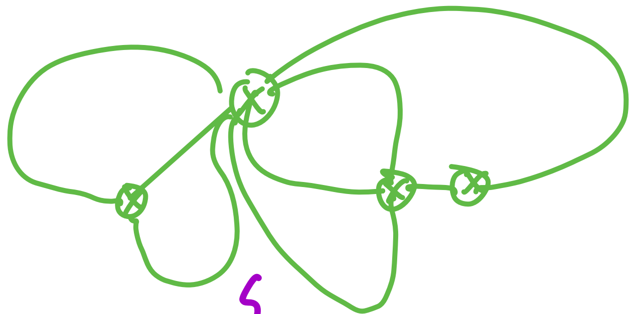
(1) G^* is always connected, and hence sometimes $(G^*)^* \not\cong G$ (if G was disconnected)



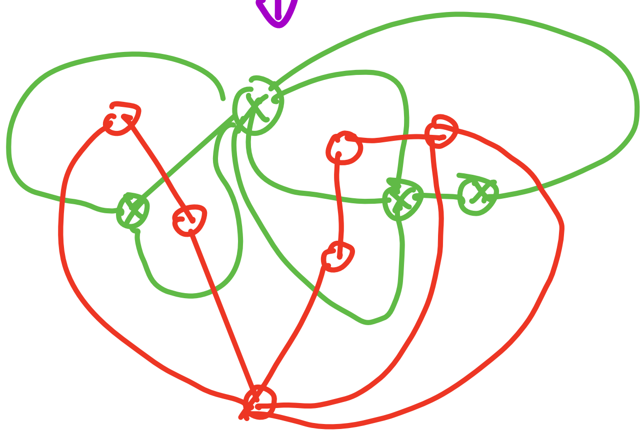
G



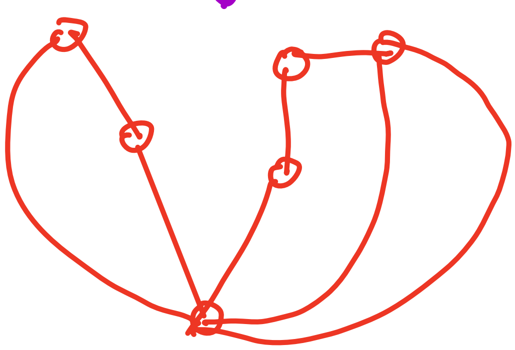
G G^*



G^*



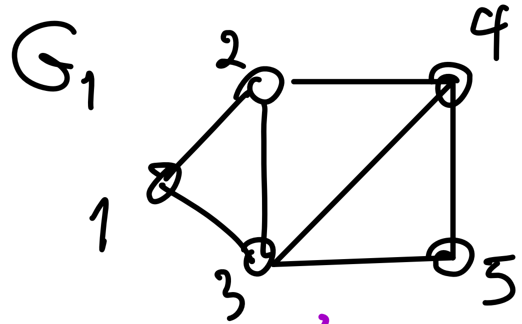
G^*



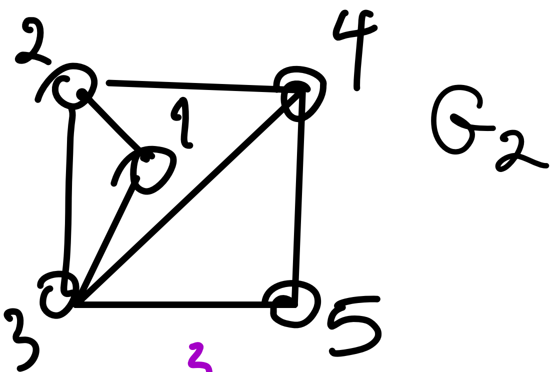
$(G^*)^* \; (\neq G)$

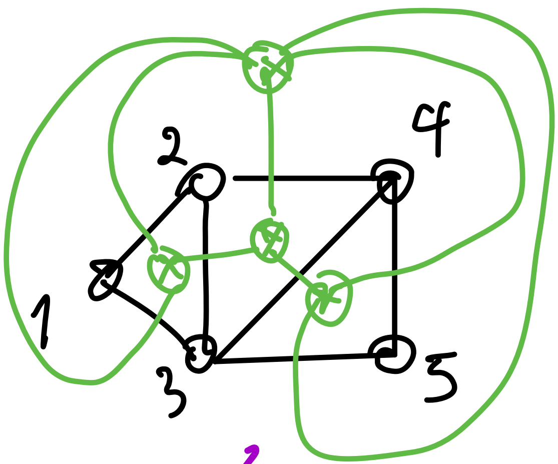
② G^* really depends on the plane embedding of G, not just on the isomorphism type of G.

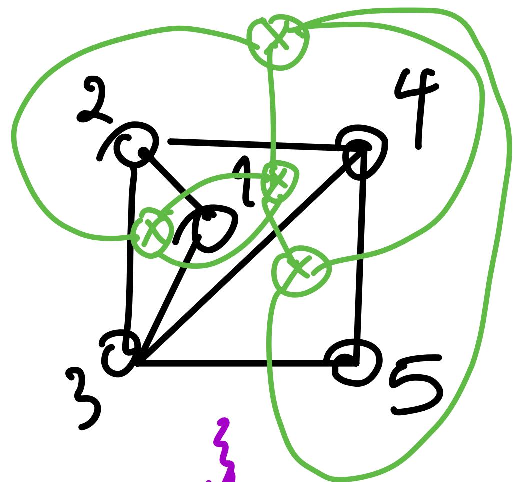
e.g.,



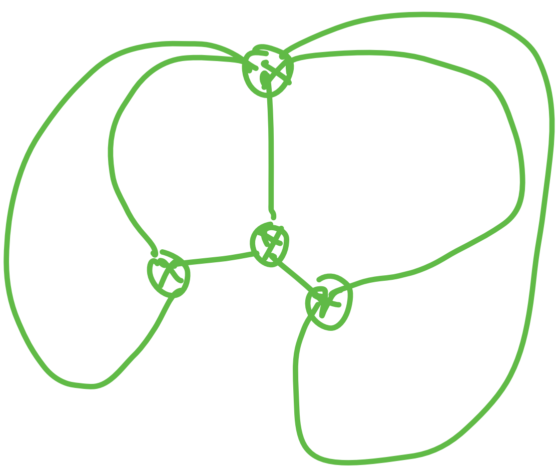
$\cong$ isomorphic



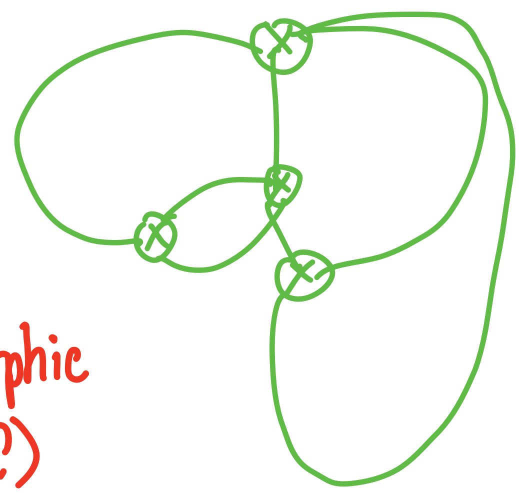




G_1^*



$\not\cong$ not isomorphic (Why?)



G_2^*

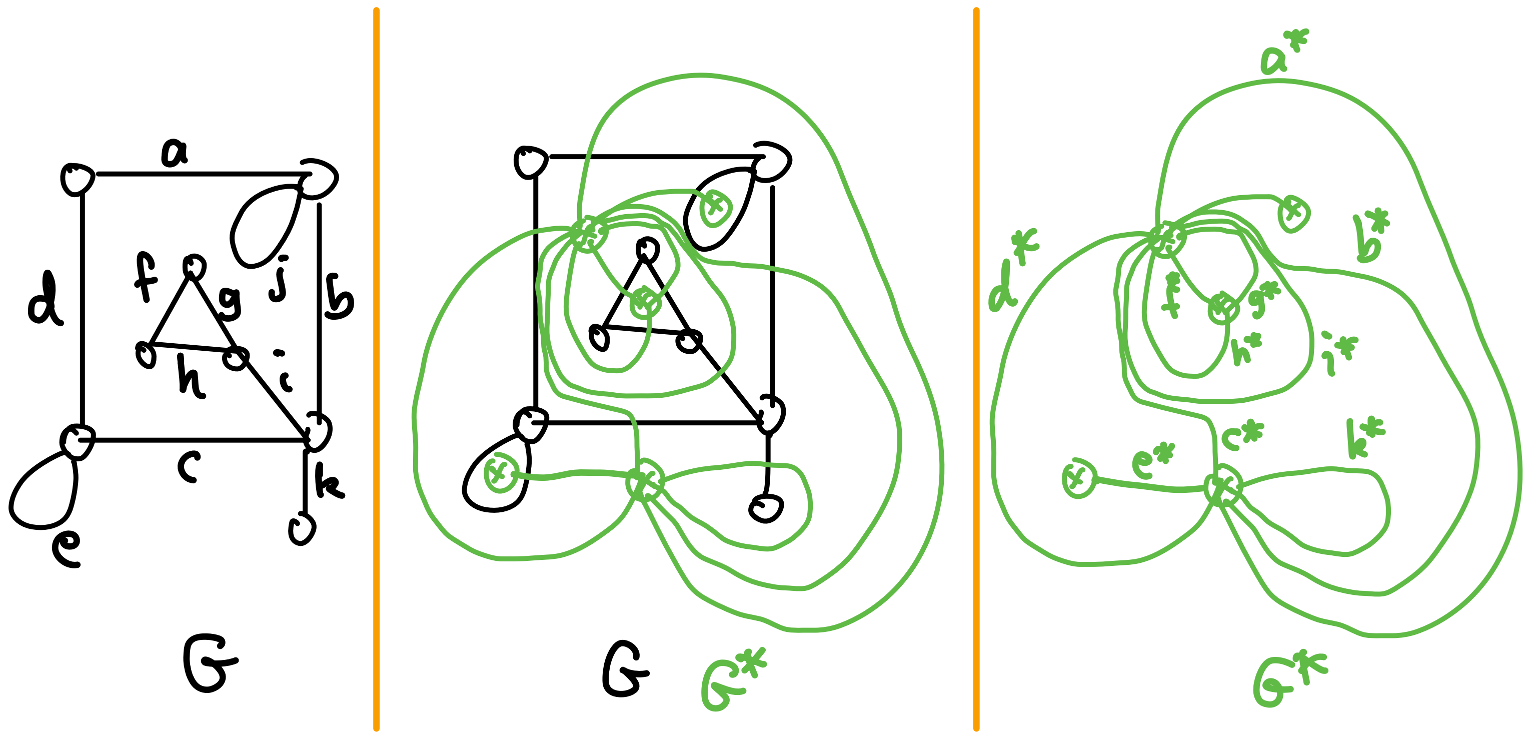
③ Even when G is simple, G^* may not be,

e.g. G_1, G_2 above are simple

but G_1^*, G_2^* have parallel edges

ACTIVE LEARNING:

In this example of G and G^*,



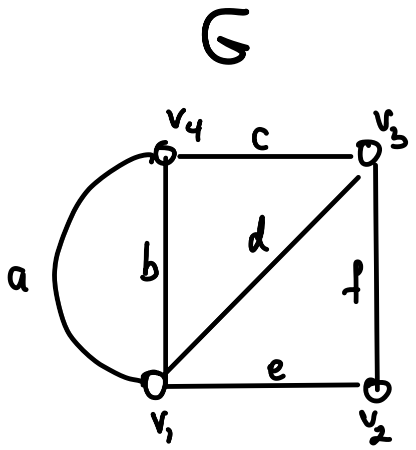
(a) who are the loop edges e in G, and what special property do their dual edges e^* have in G^*?

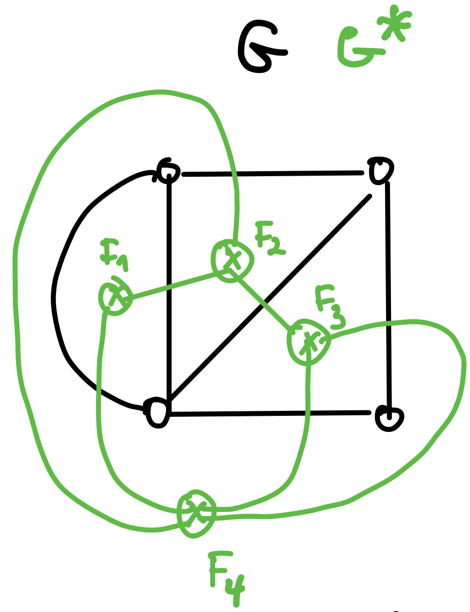
(b) who are the cut-edges e (isthmuses) in G, and what special property do their dual edges e^* have in G^*?

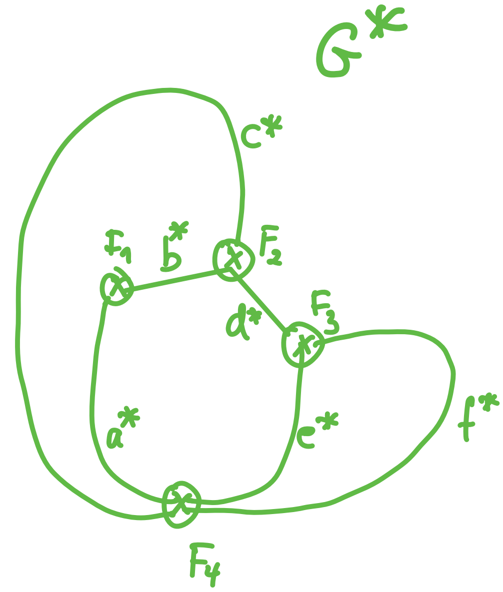
Cycles and cuts

ACTIVE LEARNING:

In this example of G and G^*,







how do these cycles C in G: relate to these partitions of the vertices $V^* = S \sqcup \bar{S}$ in G^*:

C	$(S, \bar{S})$
$\{a, c, d\}$	$(\{F_1, F_2\}, \{F_3, F_4\})$
$\{b, c, e, f\}$	$(\{F_2, F_3\}, \{F_1, F_4\})$
$\{b, c, d\}$	$(\{F_2\}, \{F_1, F_3, F_4\})$

how do these cycles C^* in G^*: relate to these partitions of $V = S \sqcup \bar{S}$ in G:

C	$(S, \bar{S})$
$\{c^*, d^*, e^*\}$	$(\{v_1, v_4\}, \{v_2, v_3\})$
$\{a^*, b^*, d^*, e^*\}$	$(\{v_1\}, \{v_2, v_3, v_4\})$

PROPOSITION:

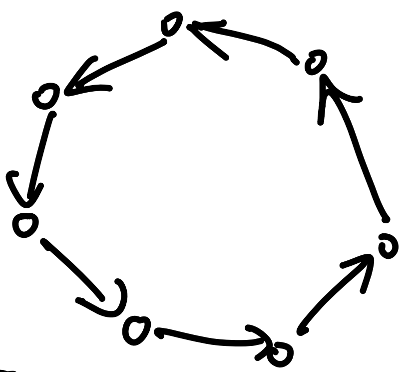
For a plane multigraph $G=(V,E)$,

$C \subseteq E$ forms a cycle in G

$\iff$ $C^* := \{e^* : e \in C\}$ forms the cut (bond) $E(S,\bar{S}) := \{e^* \in E^* : e^* = \{F,F'\}$ for some $F \in S,\ F' \in \bar{S}\}$

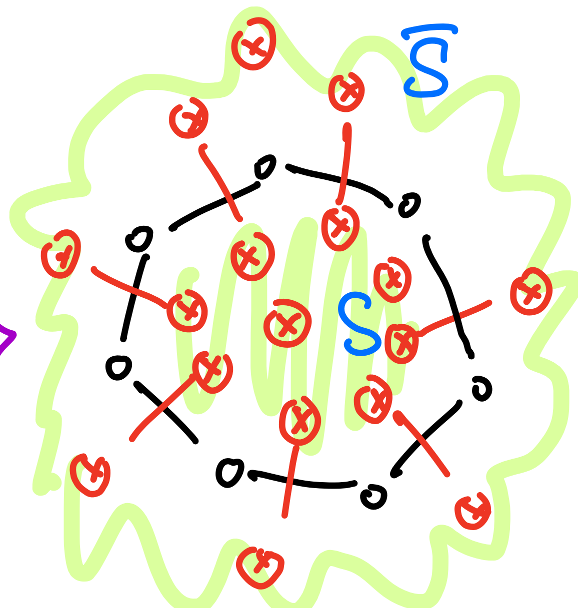
associated to some nontrivial partition $V = S \sqcup \bar{S}$

both $S \neq \emptyset$
$\bar{S} \neq \emptyset$



C in G and Ω

$\rightsquigarrow$



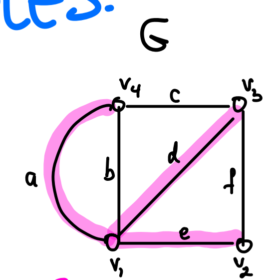
$S \leftrightsquigarrow$ faces of G inside C

$\bar{S} \leftrightsquigarrow$ faces of G outside C

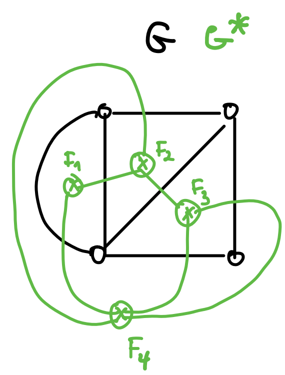
This leads to an interesting symmetry for spanning trees of G, G^*:

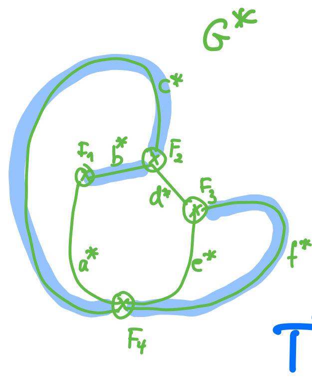
COROLLARY:
For a connected plane multigraph $G=(V,E)$,
$T \subseteq E$ is a spanning tree for G
$\iff T^* := \{e^* : e \in E \setminus T\} \subset E^*$
is a spanning tree for G^*

EXAMPLES:

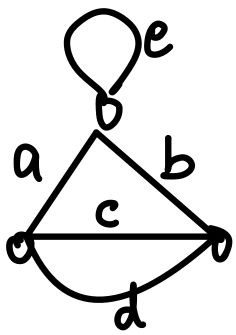
① 

$T = \{a,d,e\}$

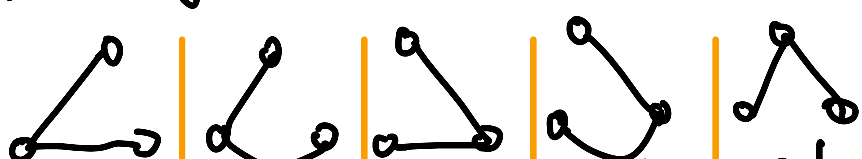




$T^* = \{b^*, c^*, f^*\}$

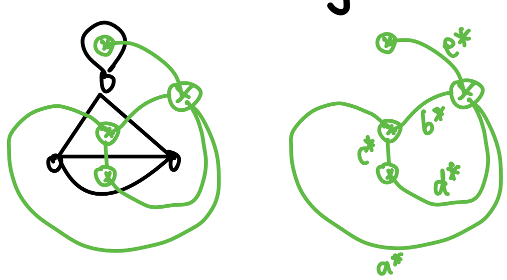
② $G=$ 

had spanning trees

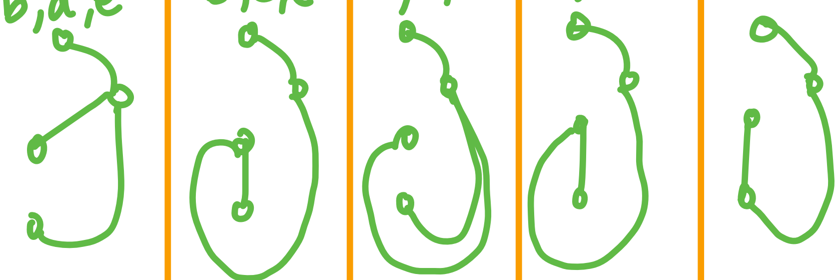


$T=$ a,c | a,d | b,c | b,d | a,b

so G^* has spanning trees



$T^* =$ b^*, d^*, e^* | b^*, c^*, e^* | a^*, d^*, e^* | a^*, c^*, e^* | c^*, d^*, e^*



proof of COROLLARY: (assuming PROPOSITION)

Because of the PROPOSITION,

T contains a cycle C in G

$\Leftrightarrow$ T^* omits all of the edges $E(S,\bar{S})$
of some nontrivial cut $V^* = S \sqcup \bar{S}$,
disconnecting S from $\bar{S}$ in G^*

$\Leftrightarrow$ T^* does not connect V^*.

Symmetrically, T does not connect G
$\Leftrightarrow$ T^* contains a cycle in G^*.

Hence T is a tree in G $\Leftrightarrow$ T^* is a tree in G^* ▨

This leads to a pleasantly symmetric
2nd proof of Euler's Formula:

Given a plane graph $G = (V, E)$ with
and plane dual $G^* = (V^*, E^*)$
with

$n = |V|$
$m = |E| = |E^*|$
$f = \#\text{faces of } G = |V^*|$,

Pick any spanning tree $T \subseteq E$ for G, so that

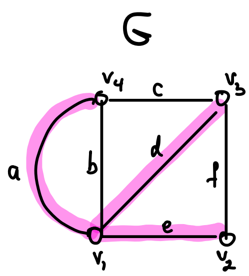
$T^* := \{e^* : e \in E \setminus T\}$ is a spanning tree for G^*.

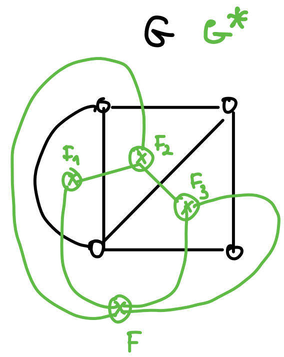
But then these two trees have $\begin{cases} |T| = |V|-1 = n-1 \\ |T^*| = |V^*|-1 = f-1 \end{cases}$

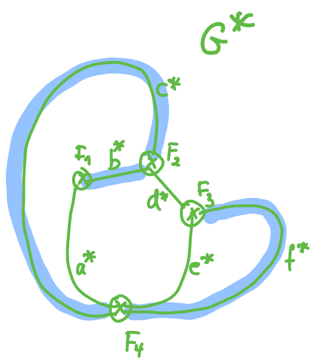
and $e = |T| + |T^*|$ by construction.

$$\text{So } e = (n-1)+(f-1)$$
$$= n+f-2$$

i.e. $\boxed{n-e+f=2}$ $\blacksquare$







$T = \{a, d, e\}$

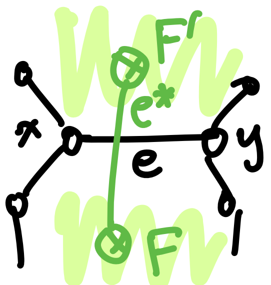
$T^* = \{b^*, c^*, f^*\}$

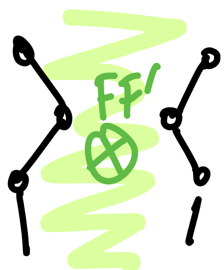
Deletion, contraction & duality

ACTIVE LEARNING: Explain why this holds:

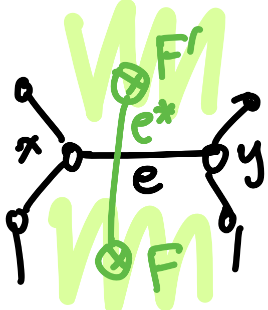
PROPOSITION: For plane dual multigraphs $\underset{(V,E)}{G}, \underset{(V^*,E^*)}{G^*}$ and an edge $e=\{x,y\}$ in G that bounds the faces F, F', one has

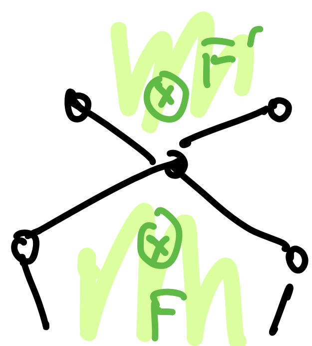
(a) $(G \setminus e)^* \cong G^*/e^*$





(b) $(G/e)^* \cong G^* \setminus e^*$



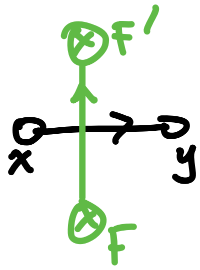


Orientations and duality

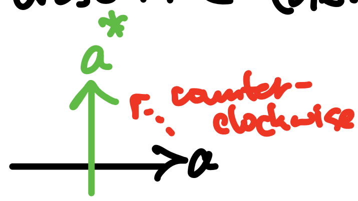
When a plane graph $G=(V,E)$ is given some orientation Ω to form a digraph $D=(V,A)$, one can also induce a corresponding orientation Ω^* to form a digraph $D^*=(V^*,A^*)$ via this rule:

if $x \underset{e}{\circ\!\!-\!\!-\!\!\circ} y$ is oriented as $x \xrightarrow{a} y$ in Ω

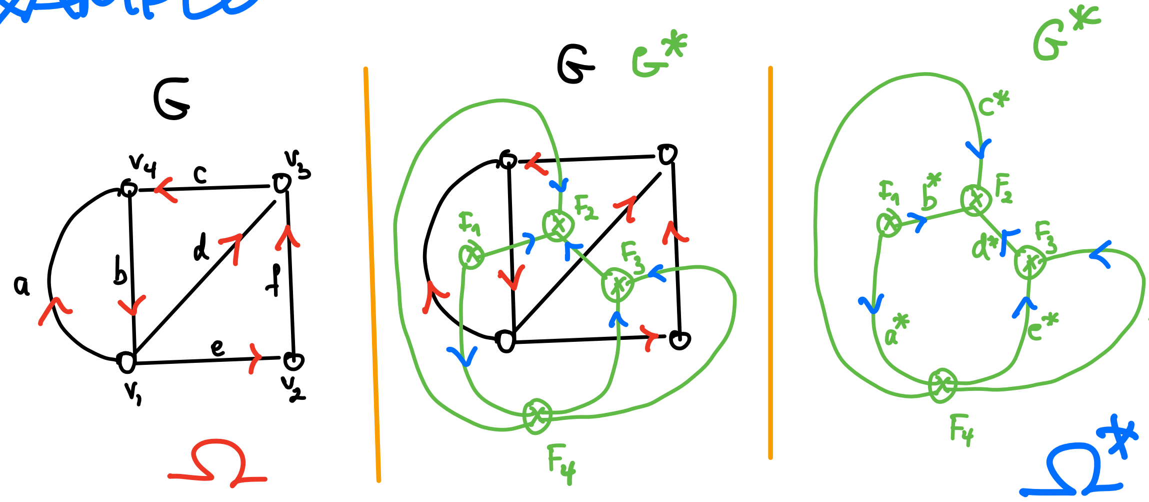
then orient e^* =



so a, a^* cross like this:

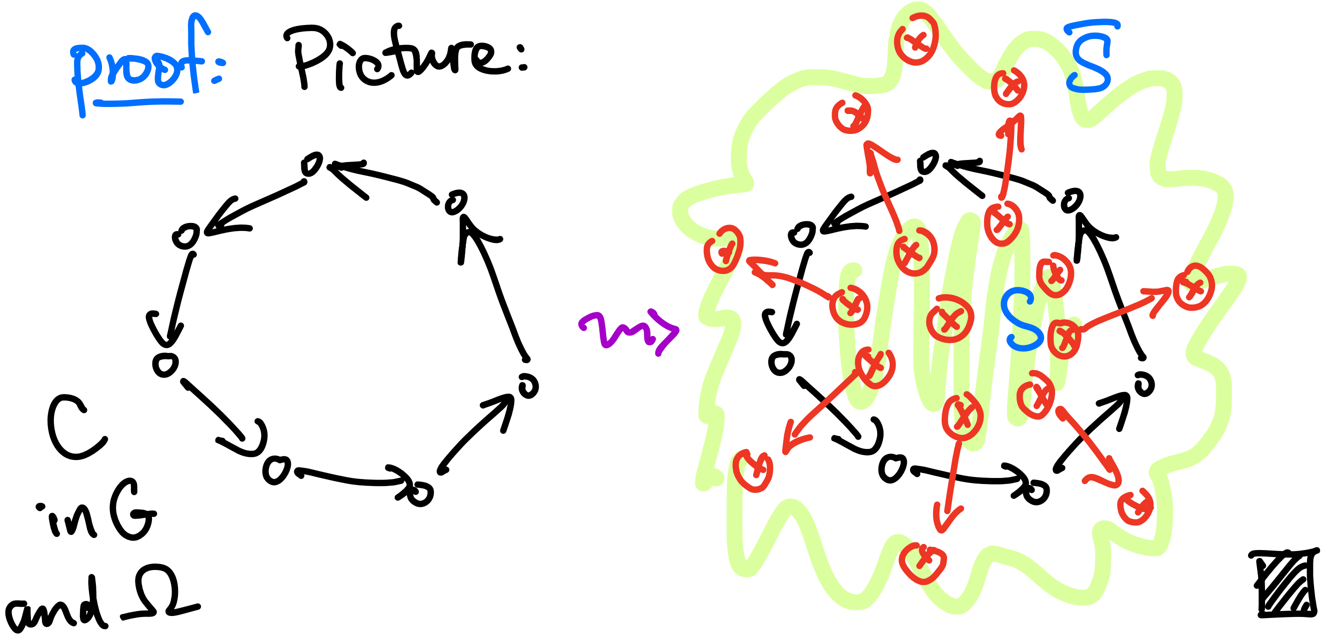


EXAMPLE



PROPOSITION:
Given a plane multigraph $G=(V,E)$ and an orientation Ω, a cycle $C\subseteq E$ forms a directed circuit in Ω
$\Leftrightarrow$ the corresponding cut $C^*=\{e^*: e\in C\}=E(S,\bar{S})$
is directed in $\hat{\Omega}$ as $A^*(S,\bar{S})=\{a^*=(F,F')\in A^*: F\in S, F'\in\bar{S}\}$
(or $A^*(\bar{S},S)$).

proof: Picture:



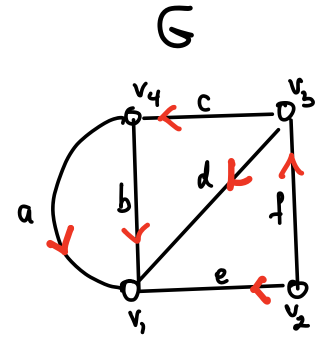
PROPOSITION: For plane dual multigraphs G, G^* with corresponding orientations Ω, Ω^*,

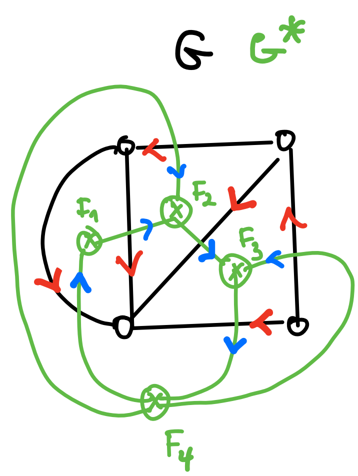
Ω is an acyclic orientation

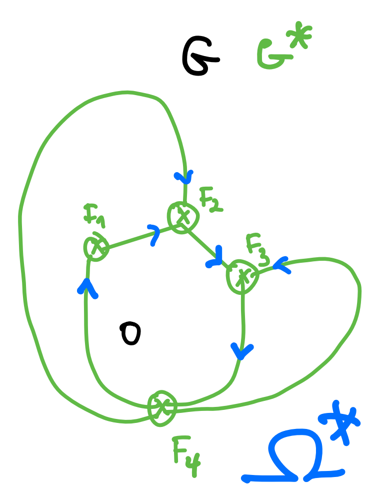
$\Leftrightarrow$ Ω^* is a totally cyclic orientation.

every arc a^* lies in at least one directed cycle

EXAMPLE







proof: If Ω is acyclic, for every edge $e = \overset{x}{\circ}\!-\!\!-\!\overset{y}{\circ}$ in G directed as $a = \overset{x}{\circ}\!\!\longrightarrow\!\!\overset{y}{\circ}$ in Ω, the sets $S = \{z \in V : \exists \text{ a path } z \to \dots \to x \text{ in } \Omega\} \ni x$

$$\bar{S} = \{z \in V : \exists \text{ a path } y \to \dots \to z \text{ in } \Omega\} \ni y$$

give a nontrivial partition $V = S \sqcup \bar{S}$, because Ω is acyclic. Furthermore, every edge going from S to $\bar{S}$ in G must be directed from S to $\bar{S}$ in Ω, again because Ω is acyclic. This means $A(S, \bar{S})$ in Ω on G is dual to a directed cycle C in Ω^* on G^* that contains e^*. So Ω^* is ~~totally~~ cyclic.

If Ω is not acyclic, a directed cycle C in Ω on G leads to a directed cut $C^* = A^*(S, \bar{S})$ in Ω^* on G^*. Since G^* is connected, $\exists$ at least one arc $a^* \in A^*(S, \bar{S})$, which cannot lie in a directed cycle in Ω^* (the cycle couldn't get back from $\bar{S}$ to S in Ω^*). ▨

Like acyclic orientations, the totally cyclically orientations have a deletion-contraction recurrence.

Let $TC(G) := \{\text{totally cyclic orientations } \Omega \text{ of } G\}$

and $tc(G) := |TC(G)|$.

PROPOSITION: For any multigraph $G = (V, E)$, $tc(G)$ can be computed via this recurrence:

- $tc(G) = 1$ if $E = \emptyset$
- $tc(G) = 0$ if G contains a cut-edge
- If e is a non-cut-edge, then $tc(G) = tc(G \setminus e) + tc(G/e)$.

[The take-home final exam for the class suggests a proof of this, similar to part of the proof earlier that $ac(G) = |\{\text{acyclic orientations of } G\}|$
$= (-1)^n \pi(G, -1)$]