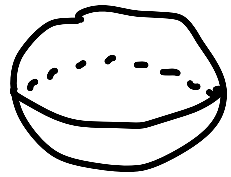
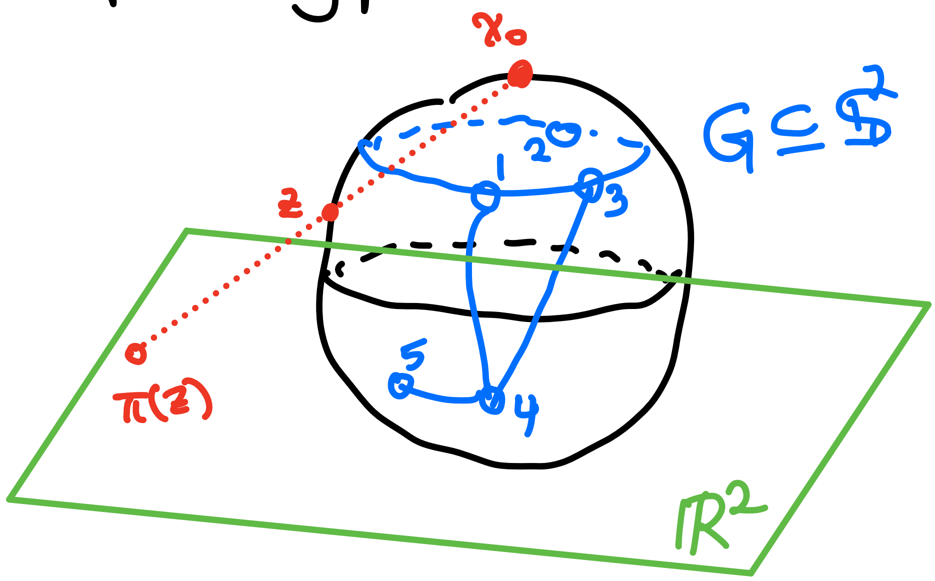
Coloring graphs on surfaces (Bollobás "Modern Graph Theory" §V.3)

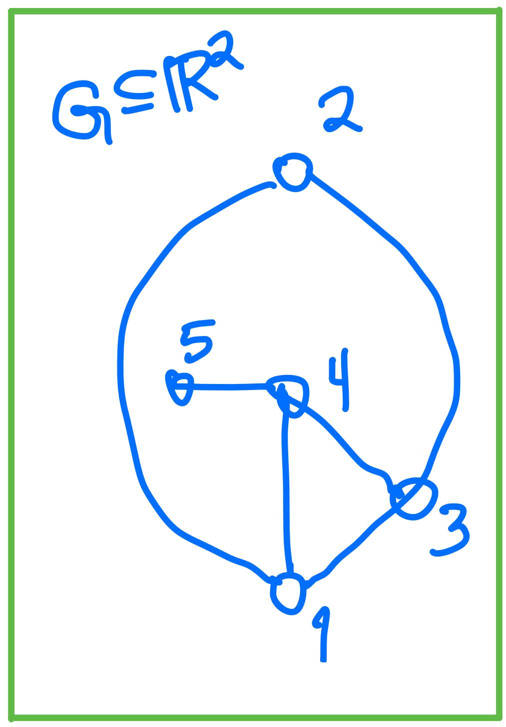
Planar multigraphs are the same as those that can be embedded in the 2-dimensional sphere $S^2 =$



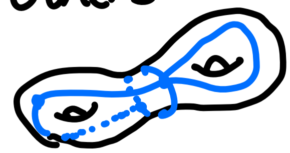
e.g. using stereographic projection $S^2 - \{x_0\} \xrightarrow{\pi} \mathbb{R}^2$

from any point $x_0 \in S^2 - G$:





In the late 1800's, P. J. Heawood (1861–1955) and others considered graphs G embedded on other



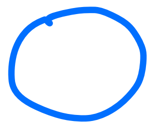
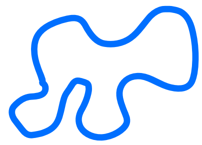
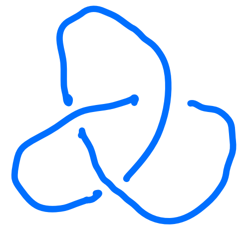
compact surfaces, orientable & non orientable

along with their vertex-colorings...

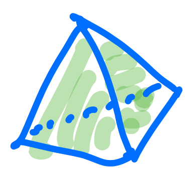
A few (rough) definitions... (from Math 5345, 8301, 8302)

DEFINITION: Two topological spaces X, Y are **homeomorphic** ($X \cong Y$) if $\exists$ a bijection $X \underset{f^{-1}}{\overset{f}{\rightleftarrows}} Y$ with both f, f^{-1} continuous maps.

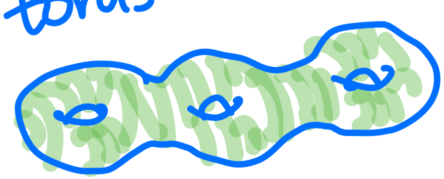
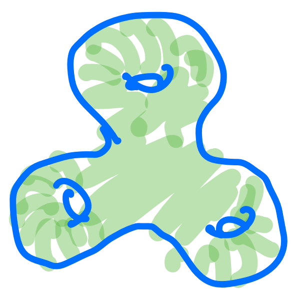
e.g.

$S^1 =$  $\cong$  $\cong$ 

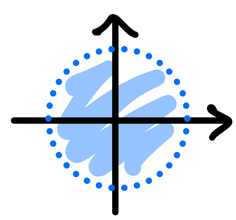
= circle = 1-dimensional sphere

$S^2 =$  $\cong$ 

3-holed torus

 $\cong$ 

A subset $S \subset \mathbb{R}^d$ is a surface (= 2-manifold) if every $x \in S$ has a neighborhood homeomorphic to the open disk $x^2+y^2<1$ in $\mathbb{R}^2$

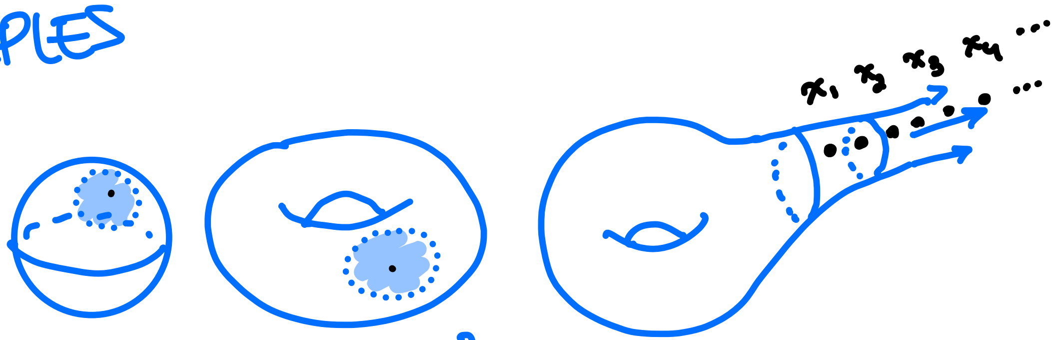


S is compact if every infinite sequence $x_1, x_2, \dots$ in S has a convergent subsequence $x_{i_1}, x_{i_2}, \dots \to x_\infty$ in S; equivalently S is closed and bounded in $\mathbb{R}^d$

$i_1 < i_2 < \dots$

EXAMPLES

①



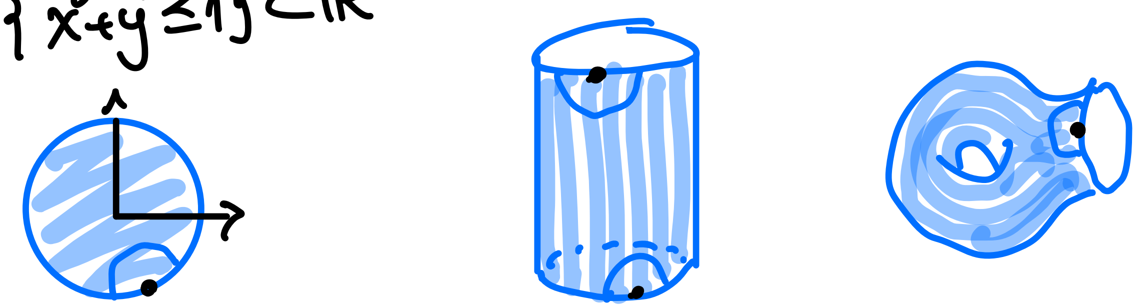
S^2 torus T^2 not compact

compact surfaces

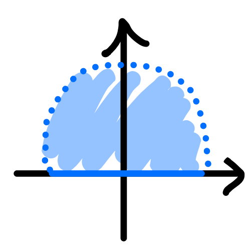
all are surfaces

② closed disk

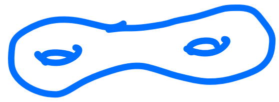
$\{x^2+y^2 \le 1\} \subset \mathbb{R}^2$



are all compact, but not quite surfaces due to boundary points x where neighborhood $\cong$



DEFINITION: A compact surface S is orientable if it embeds in $\mathbb{R}^3$



THEOREM (Jordan-Brouwer Separation) Every orientable surface S divides $\mathbb{R}^3 - S$ into two connected components,

one bounded (the interior of S)

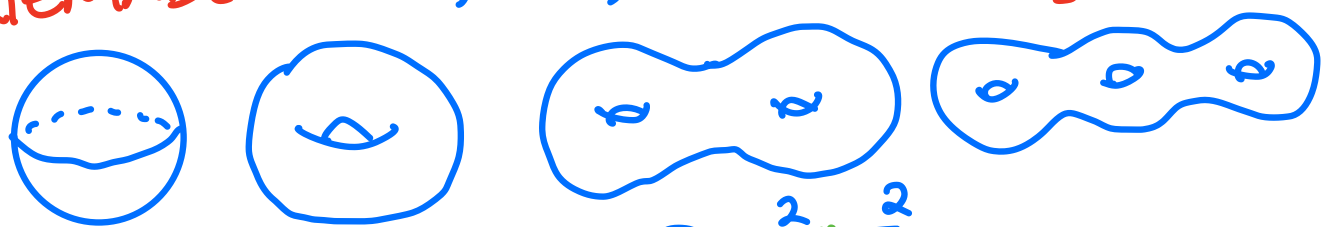
one unbounded (the exterior of S)

THEOREM: (Riemann, 1851, Poincaré and others late 1800's)

(takes work!)

There are two families of connected compact surfaces:

- ORIENTABLES: $S_0, S_1, S_2, S_3, \dots$ $\quad S_g :=$ torus with g handles



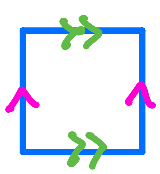
$S_0 := S^2 \quad S_1 = T^2 =$ 2-torus $\quad S_2 = T^2 \# T^2$

connected sum

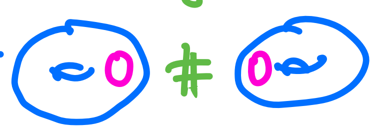
$=$

glue

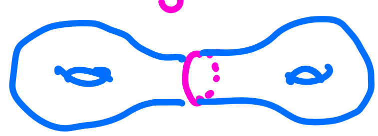
$=$



$=$



glue



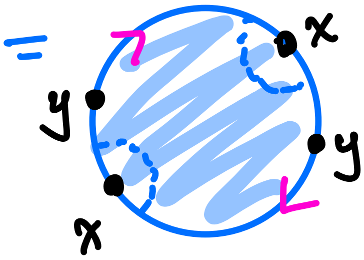
- NON-ORIENTABLES $N_1, N_2, N_3, \ldots$
(which don't embed in $\mathbb{R}^3$, but do embed in $\mathbb{R}^4$)

$N_1 = P^2_{\mathbb{R}} = P^2$ = real projective plane

$N_2 = P^2 \# P^2$ = Klein bottle

$N_3 = P^2 \# P^2 \# P^2$

=



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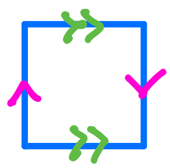

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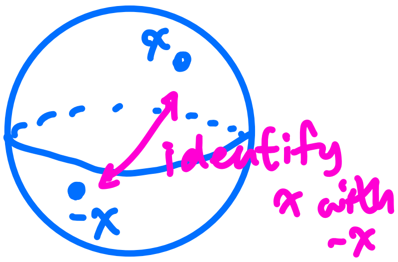

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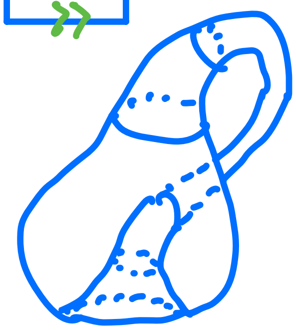


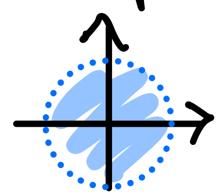
REMARK: This classification of surfaces (2-manifolds) should be compared with the simpler classification of connected 1-manifolds:

	compact	non-compact
1-manifold	S^1	$\mathbb{R}^1$
1-manifold with boundary	$[0,1]$	$[0,+\infty)$

In thinking about the topology of graphs G embedded on surfaces and vertex-colorings of G, they realized it sometimes helps to impose constraints.

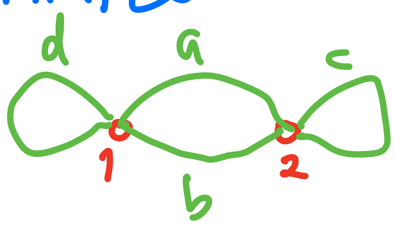
DEFINITION: An embedding of a multigraph $G=(V,E)$ on a surface S is called a **2-cell embedding** if every connected component of $S \setminus G$ is homeomorphic to the open disk

$x^2+y^2<1$

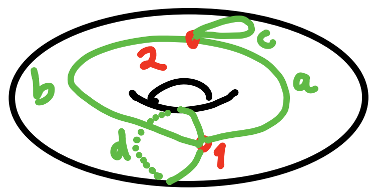


EXAMPLE

$G=$

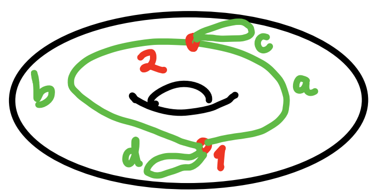


embedded as



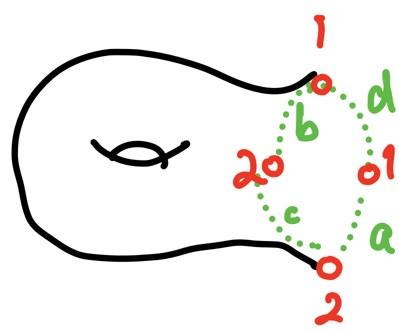
in T^2 is a 2-cell embedding

but embedded in T^2 as



it is not:

$T^2 \setminus G$ has 3 components,
one inside c,
one inside d,
and one homeomorphic to this:



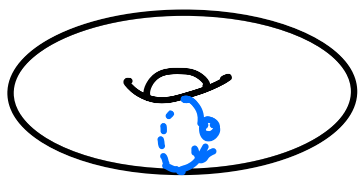
THEOREM: (not hard) (a) Any embedding of a multigraph G on a compact surface S can be made into a 2-cell embedding by adding more edges (but no new vertices).

(b) It can even be made into a 2-cell embedding where each component of $S \setminus G$ is bounded by ≤ 3 edges.

(c) In particular, if G is simple, it can be made into a triangulation of S.
(every component of $S \setminus G$ bounded by exactly 3 edges)

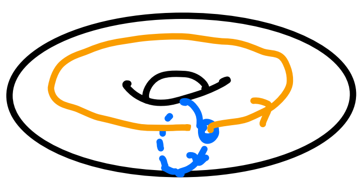
EXAMPLES:

① $G =$ 

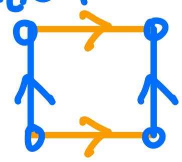


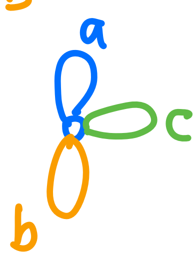
not a 2-cell embedding

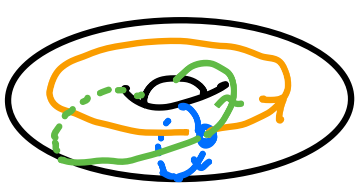
$G^{+} =$ 



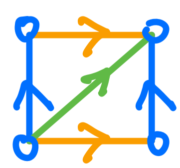
a 2-cell embedding, not a triangulation



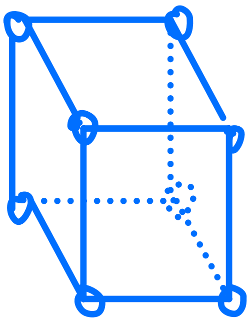
$G^{++} =$ 

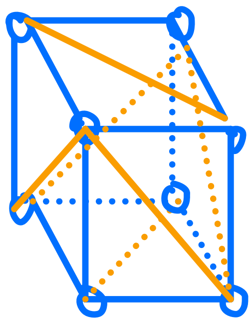


a triangulation



②

$G =$  a 2-cell embedding on S^2, not a triangulation

$G^t =$  a triangulation

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THEOREM: $\exists$ a homeomorphism invariant $\chi(S)$ of a compact surface S, called its Euler characteristic, such that every 2-cell embedding of any graph $G=(V,E)$ on S
(relatively sophisticated)

has $\boxed{n - m + f = \chi(S)}$

where $n := |V|$

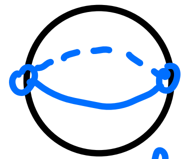
$m := |E|$

$f :=$ # of faces $=$ 2-cell components of $S \setminus G$

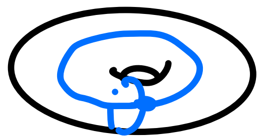
EXAMPLES

① $\chi(S_0) = 2$, $\chi(S_1) = 0$, $\chi(N_1) = 1$

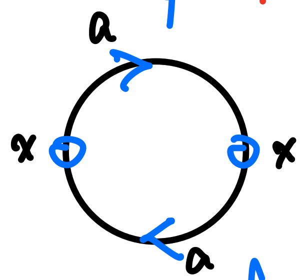
$S_0 = $ sphere S^2 $\quad$ $S_1 = $ torus T^2 $\quad$ $N_1 = P^2$ projective plane



$n - m + f = 2 - 2 + 2 = 2$



$n - m + f = 1 - 2 + 1 = 0$



$n - m + f = 1 - 1 + 1 = 1$

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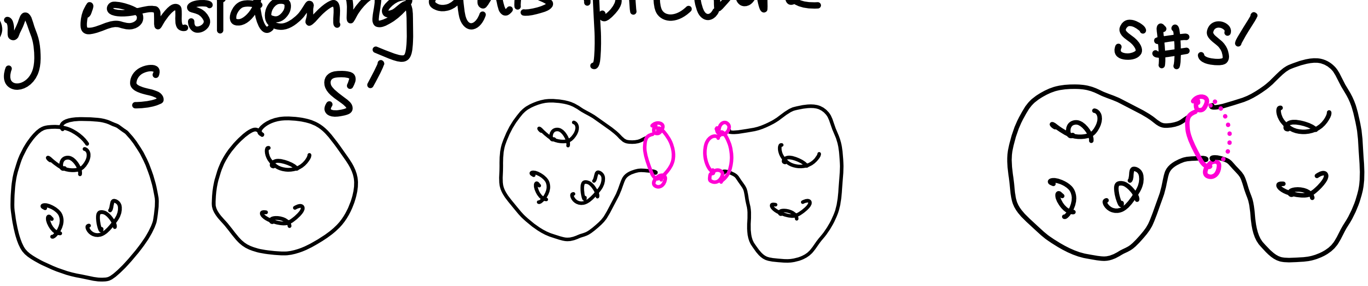
② ACTIVE LEARNING:

(a) Prove this:

$$\boxed{\text{PROPOSITION: } \chi(S \# S') = \chi(S) + \chi(S') - 2}$$

by considering this picture:



(b) Deduce that $\chi(S_g) = 2 - 2g$ for $g = 0, 1, 2, \ldots$

$$S_g = \underbrace{T^2 \# T^2 \# \ldots \# T^2}_{g \text{ times}}$$

(c) ... and that $\chi(N_g) = 2 - g$ for $g = 1, 2, 3, \ldots$

$$N_g = \underbrace{P^2 \# P^2 \# \ldots \# \mathbb{P}^2}_{g \text{ times}}$$

COROLLARY: If G is a simple graph with a 2-cell embedding on a compact surface S, then $m \leq 3(n - \chi(S))$

with equality $\Leftrightarrow$ it is a triangulation.

proof: Just slightly modify our proof from the planar graph discussion. As before, show $\boxed{2m \geq 3f}$ with equality $\Leftrightarrow$ triangulation:

$$\underbrace{\sum_{e \in E} 2}_{= 2m} = \left| \left\{ (e, F) : \begin{array}{l} F \text{ a (2-cell) face,} \\ e \text{ an edge bounding } F \end{array} \right\} \right|$$

$$= \sum_{\text{faces } F} \underbrace{\left| \left\{ \begin{array}{l} \text{edges } e \\ \text{bounding } F \end{array} \right\} \right|}_{\substack{\geq 3 \\ \text{since } G \text{ is simple}}}$$

$$\geq 3f.$$

with equality $\Leftrightarrow$ triangulation

So $f \leq \frac{2}{3} m$, and then use the Euler characteristic relation

$$n - m + f = \chi(S)$$

$$\Rightarrow \quad n - m + \frac{2}{3} m \geq \chi(S)$$

$$n - \frac{m}{3} \geq \chi(S)$$

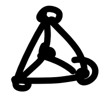
$$3n - m \geq 3\chi(S)$$

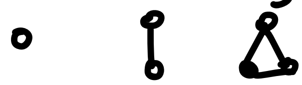
$$m \leq 3(n - \chi(S)) \quad ▨$$

This constrains which complete graphs K_n can be embedded on surface S, in terms of $\chi(S)$.

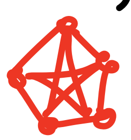
EXAMPLES:

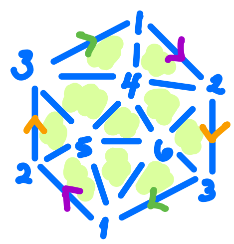
① We saw on $S_0 = \mathbb{S}^2 = $ 2-sphere  with $\chi(S_0) = 2$,

one can embed $K_1 \subset K_2 \subset K_3 \subset K_4 =$ 



but not K_5, since it has $m = \binom{5}{2} = 10 \not\leq 3(n-2) = 3(5-2) = 9$

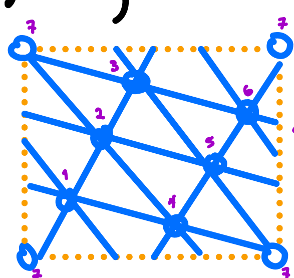


② On $N_1 = P^2 =$ projective plane, with $\chi(N_1) = 1$, one can embed $K_1 \subset K_2 \subset K_3 \subset K_4 \subset K_5 \subset K_6 =$ 

minimal triangulation of P^2

(= icosahedron boundary identified via $x \mapsto -x$)

but not K_7, since it has $m = \binom{7}{2} = 21 \not\leq 3(n-1) = 3(7-1) = 18$

③ On $S_1 = T^2 =$ 2-torus , with $\chi(S_1) = 0$, one can embed $K_1 \subset K_2 \subset K_3 \subset K_4 \subset K_5 \subset K_6 \subset K_7 =$ 

Möbius torus (1861)

but not K_8, since it has $m = \binom{7}{2} = 21 \not\leq 3(n-0) = 3(8-0) = 24$

These are examples of the following:

DEFINITION: For a compact surface S, the Heawood number $h(S)$ is the upper bound on n in K_n for it to have a chance of triangulating S, i.e.

$$h(S) := \max\left\{ n \in \{1,2,3,\dots\} : \binom{n}{2} \leq 3(n - \chi(S)) \right\}$$

$\binom{n}{2} = m$ for K_n

$$\Updownarrow$$

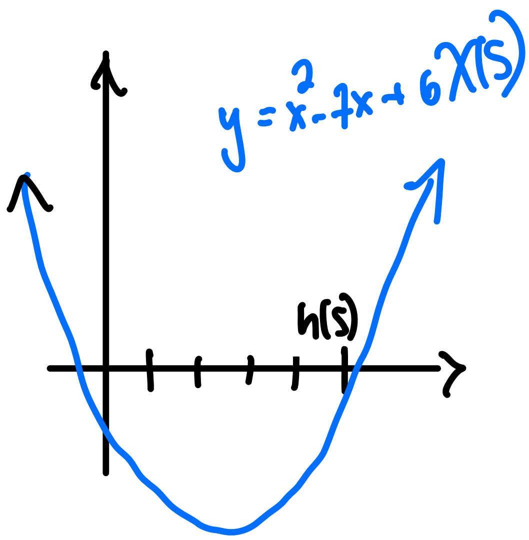
$$\frac{n(n-1)}{2} - 3n + 3\chi(S) \leq 0$$

$$\Updownarrow$$

$$n^2 - 7n + 6\chi(S) \leq 0$$

$$\Updownarrow$$

$$n \leq \frac{7 + \sqrt{7^2 - 4 \cdot 6\chi(S)}}{2}$$



that is,

$$\boxed{h(S) := \left\lfloor \frac{7 \pm \sqrt{49 - 24\chi(S)}}{2} \right\rfloor}$$

	$\chi(S)$	$h(S)$
$S_0 = S^2$	2	$\lfloor 4 \rfloor = 4$
$N_1 = P^2$	1	$\lfloor 6 \rfloor = 6$
$S_1 = T^2$, $N_2 =$ Klein bottle	0	$\lfloor 7 \rfloor = 7$
N_3	-1	$\left\lfloor \frac{7+\sqrt{73}}{2} \right\rfloor = 7$
N_3, $S_2 =$	-2	$\left\lfloor \frac{7+\sqrt{97}}{2} \right\rfloor = 8$

THEOREM (Heawood 1890)

For a graph G embedded on any surface S with $\chi(S) \leq 1$, the chromatic number of G is at most $h(S)$.

excludes the tough case $\chi(S)=2$ i.e. $S=S^2$ or planar graphs, even though $h(S)=4$!

proof: By induction on $n=|V|$. We'll show that the idea underlying the proof of the 6-color theorem works whenever $\chi(S) \leq 1$.

BASE CASE: $n \leq h(S)$.

No problem; give each vertex a different color.

INDUCTIVE STEP: $n \geq h(S)+1$.

Without loss of generality, G is simple and 2-cell embedded on S, so

$$|E| = m \leq 3(n-\chi(S))$$

and

$$\text{average degree in } G = \frac{1}{n}\sum_{x \in V} \deg_G(x) = \frac{2|E|}{n} = \frac{2m}{n}$$

$$\leq \frac{6(n-\chi(S))}{n} = 6 - 6\frac{\chi(S)}{n}$$

We'll show that this always forces existence of a vertex x_0 of degree $\le h(\chi)-1$, since then by induction, $G-\{x_0\}$ has a proper $h(\chi)$-coloring, which extends on x_0 to one for G.

CASE 1: $\chi(S)=1$, so $h(S)=6$

Then $\underset{\text{in } G}{\overset{\text{average}}{\text{degree}}} \le 6-6\frac{\chi(S)}{n} < 6$

so $\exists$ a vertex of degree $\le 5 = h(\chi)-1$, as desired.

CASE 2: $\chi(S) \in \{0,-1,-2,\dots\}$ i.e. $\chi(S)\le 0$.

If there is no vertex of degree $< h(S)$, we'll reach a contradiction, since it implies

$$h(S) \le \underset{\text{in } G}{\overset{\text{average}}{\text{degree}}} \le 6-6\frac{\chi(S)}{n} \le 6-\frac{6\chi(S)}{h(S)+1}$$

because $\chi(S)\le 0$ and $n \ge h(S)+1$

i.e., $h \le 6-\frac{6\chi(S)}{h+1}$ where $h := h(S)$

multiply by $h+1$

$$h(h+1) \le 6h+6-6\chi(S)$$

$$h^2-5h+6\chi(S)-6 \le 0$$

$$\Rightarrow h \le \frac{5+\sqrt{5^2-4(6\chi(S)-6)}}{2} = \frac{5+\sqrt{49-24\chi(S)}}{2}$$

$$= \frac{7+\sqrt{49-24\chi(S)}}{2}-1 = h(S)-1.$$

Contradiction ∎

Heawood also conjectured his theorem was tight, in the sense that for every compact surface S, there is at least one graph G embedded on S with chromatic number $h(S)$.

This turned out to be not quite right, but close:

THEOREM (Franklin 1930)

Graphs G which are 2-cell embedded on the Klein bottle $N_2 = P^2 \# P^2$, having $\chi(N_2)=0$ and Heawood number $h(N_2)=7$, actually have chromatic number at most 6 (including $G=K_6$, having chromatic number exactly 6).

THEOREM (Ringels & Youngs 1954–1970)

many papers, treating various S, orientable, non-orientable

For every compact surface $S \neq N_2 = \text{Klein bottle}$, one can give a 2-cell embedding of the complete graph $K_{h(S)}$ on S.

REMARK:

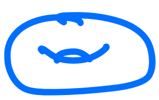
The class of graphs embeddable on a surface S is easily seen to be closed under taking minors

(= vertex-deletions,
edge-deletions,
edge-contractions).

Hence the GRAPH MINORS THEOREM of Robertson & Seymour implies this class is always characterized by a finite list of forbidden minors $\{G_1, G_2, \ldots, G_r\}$.

This list is known only for two surfaces:

- for $S = \mathbb{S}^2$, i.e. planar graphs, they avoid the minors $\{K_5, K_{3,3}\}$
- for $S = \mathbb{P}^2$, i.e. projective planar graphs, they avoid a list of 35 graphs found by Glover, Huneke & Wang 1979

For the torus S:  people expect thousands, and for the Klein bottle N_2, tens of thousands (!)