Math 5707: Graph Theory

INTRO Day 1

- Go over syllabus items, arrange office hours

e.g. 6 HW's 40%
Exam 1 20%
Exam 2 20%
Final 20%
} take-home

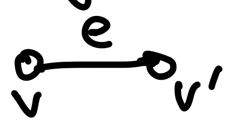
- free (old) text by Bondy & Murty

doing much of Chapters 1–5, 8, 9, 11 + some other material.

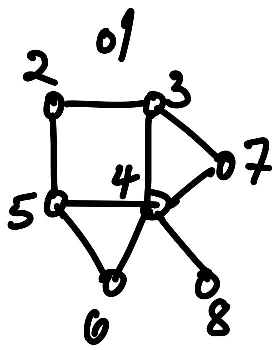
- Higher level, less counting than Math 4707
- Some optimization as in Math 5711 / IE 5531

DEFINITION: A simple graph $G = (V, E)$
V = vertices, E = edges

has a vertex set V and an edge set E (none repeated) where each edge $e \in E$ is a pair $\{v, v'\}$ of vertices $v, v' \in V$. (with $v \neq v'$)



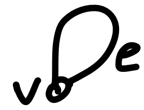
EXAMPLE

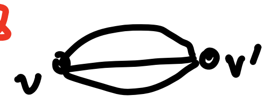
$G =$ 

$= (V, E)$

$V = \{1,2,3,4,5,6,7,8\}$

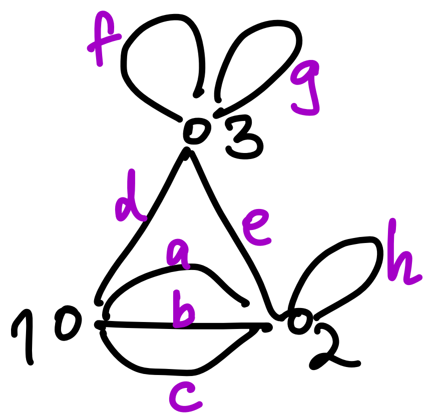
$E = \{\{2,3\}, \{3,4\}, \{4,5\}, \{5,2\}, \{3,7\}, \{4,7\}, \{5,6\}, \{4,6\}, \{4,8\}\}$

DEF'N: A multigraph $G=(V,E)$ allows E to contain loops (= self-loops) 

and parallel (= multiple, repeated) edges 

Bondy-Murty calls this just a graph

EXAMPLE

$G =$ 

$= (V, E)$

$V = \{1,2,3\}$

$E = \{a, b, c, d, e, f, g, h\}$

$a = \{1,2\}$, $b = \{1,2\}$, $c = \{1,2\}$, $d = \{1,3\}$, $e = \{2,3\}$, $f = \{3,3\}$, $g = \{3,3\}$, $h = \{2,2\}$

Bondy-Murty uses notations

$\nu(G) = |V| = \#$ of vertices in G

$\epsilon(G) = |E| = \#$ of edges in G

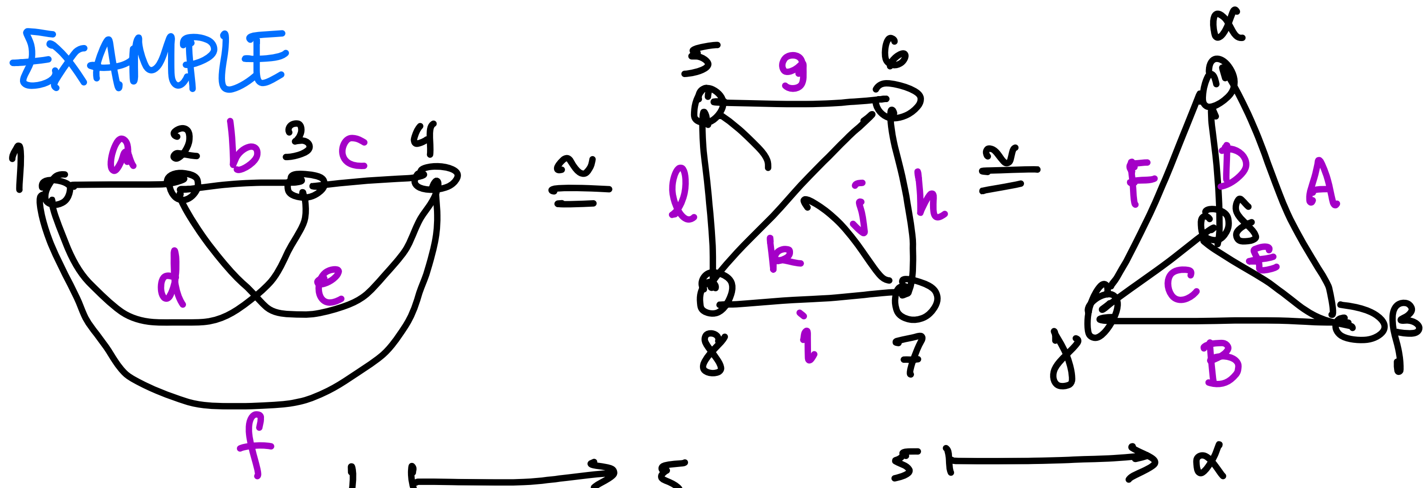
We'll mainly care about properties of a graph G that don't depend on the labeling sets for V, E; called isomorphism invariants

DEF'N: Say $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ two graphs are **isomorphic**, written $G_1 \cong G_2$, if there are **bijections** (= injection + surjection) $f: V_1 \longrightarrow V_2$, $g: E_1 \longrightarrow E_2$

such that $\forall e = \{v, v'\} \in E_1$

one has $g(e) = \{f(v), f(v')\} \in E_2$

EXAMPLE



$1 \longmapsto 5$
$2 \longmapsto 6$
$3 \longmapsto 7$
$4 \longmapsto 8$

$5 \longmapsto \alpha$
$6 \longmapsto \beta$
$7 \longmapsto \gamma$
$8 \longmapsto \delta$

$a \longmapsto g$
$b \longmapsto h$
$c \longmapsto i$
$d \longmapsto j$
$e \longmapsto k$
$f \longmapsto l$

$g \longmapsto A$
$h \longmapsto B$
$i \longmapsto C$
$j \longmapsto D$
$k \longmapsto E$
$l \longmapsto F$

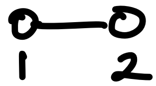
Some important EXAMPLES

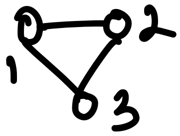
① Complete graphs

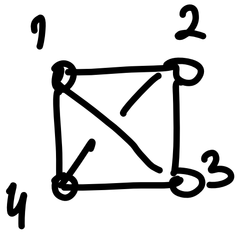
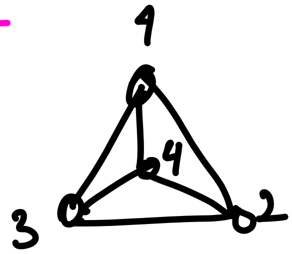
$K_n = (V, E)$

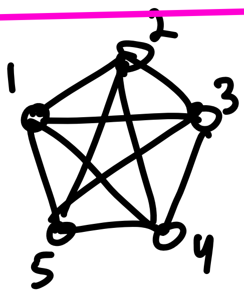
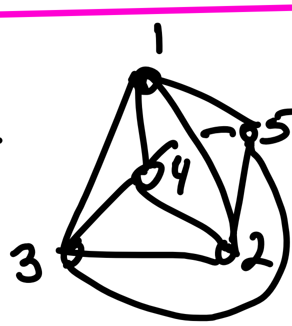
$V = \{1, 2, \ldots, n\}$, $E = \{$all pairs $\{i, j\}$ with $1 \le i < j \le n\}$

e.g. $K_1 =$ 

$K_2 =$ 

$K_3 =$ 

$K_4 =$  $\cong$ 

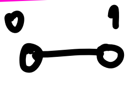
$K_5 =$  $\cong$ 

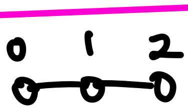
② Paths

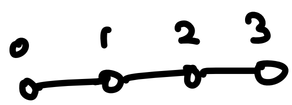
$P_m = (V, E)$ with m edges for $m \ge 0$

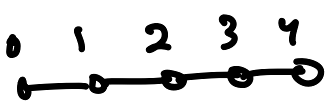
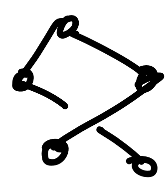
$V = \{0, 1, 2, \ldots, m\}$, $E = \{\{0, 1\}$

$P_0 =$ 

$P_1 =$ 

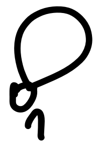
$P_2 =$ 

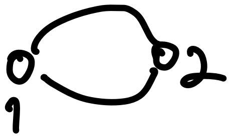
$P_3 =$ 

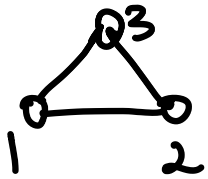
$P_4 =$  $\cong$ 

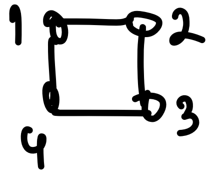
③ Cycles C_m with m edges for $m \geq 1$

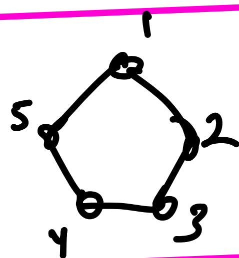
$$C_m = (V, E), \quad V = \{1,2,\dots,m\}, \quad E = \{\{1,2\},\{2,3\},\dots,\{m-1,m\},\{m,1\}\}$$

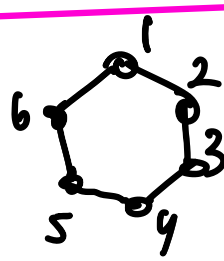
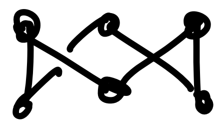
C_1 

C_2 

C_3  $= K_3$

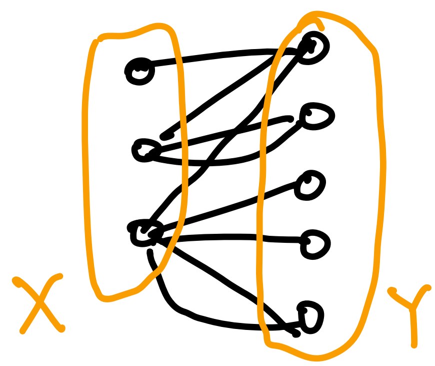
C_4 

C_5 

C_6  $\cong$ 

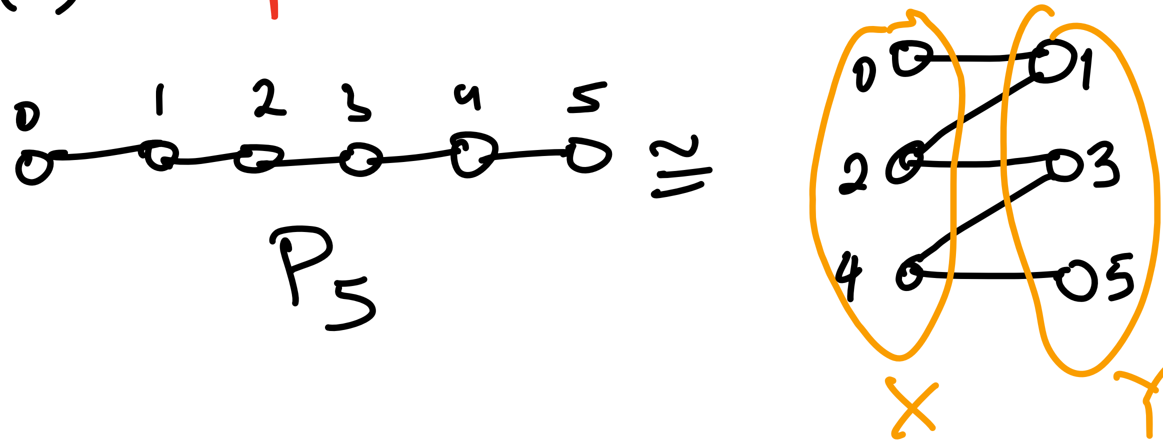
DEF'N: A graph $G=(V,E)$ is called **bipartite**, with vertex bipartition $V = X \sqcup Y$ if every edge $e \in E$ has $e = \{x, y\}$ for some $x \in X$, $y \in Y$

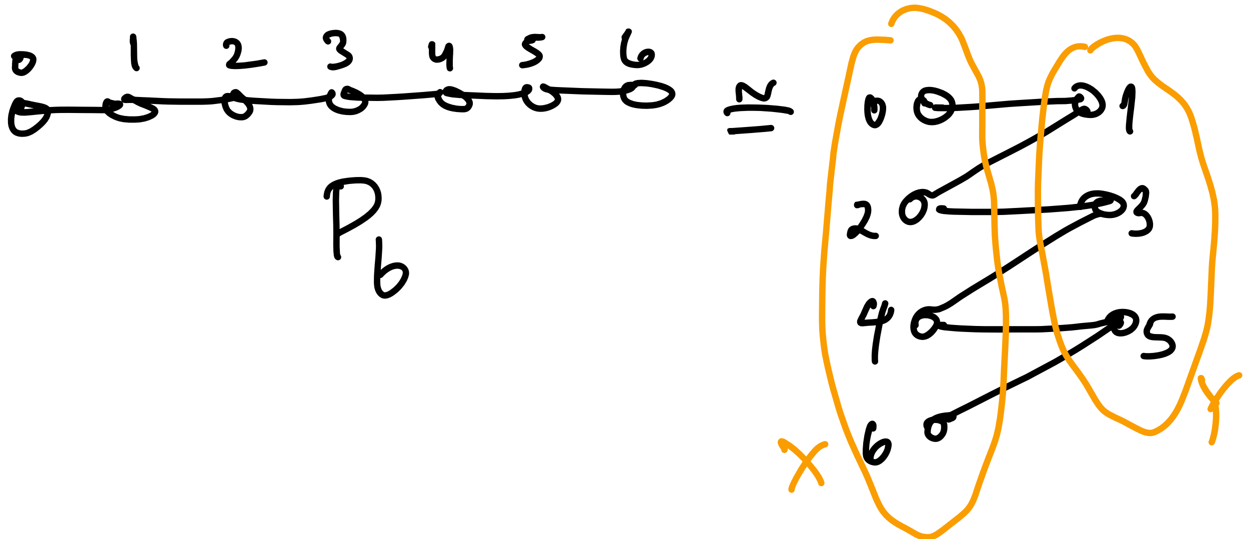
disjoint union: $V = X \cup Y$ and $X \cap Y = \phi$



PROPOSITION: (easy-EXERCISE!)

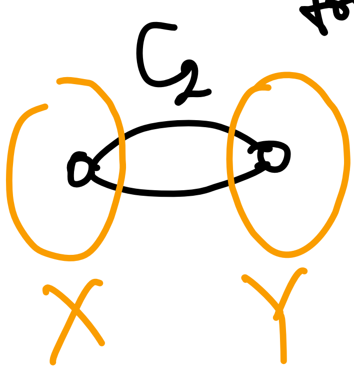
(a) all paths P_m for $m \geq 0$ are bipartite

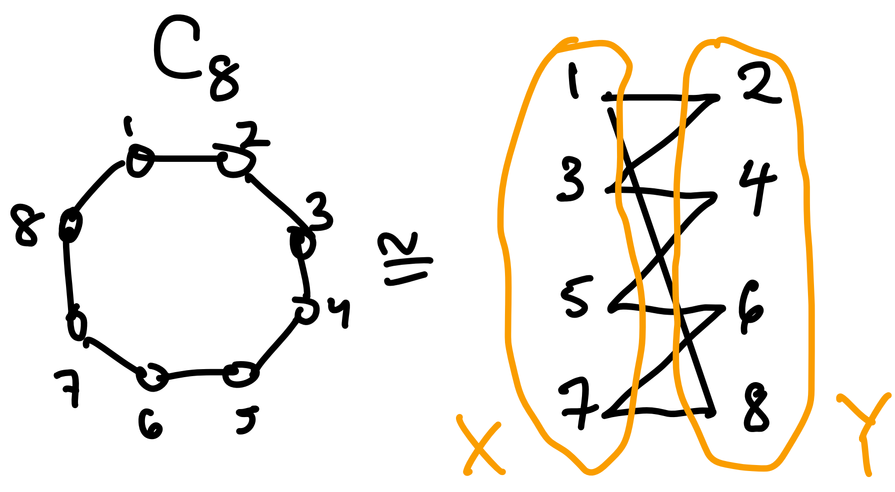




(b) cycles C_m are bipartite $\iff$ m is even

for $m \geq 1$





Continuing EXAMPLES ...

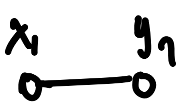
④ Complete bipartite graphs

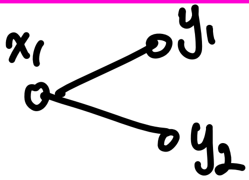
$$K_{m,n} = (V, E)$$

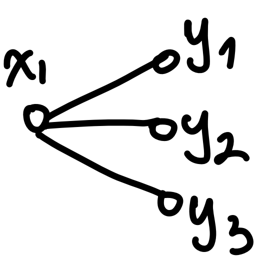
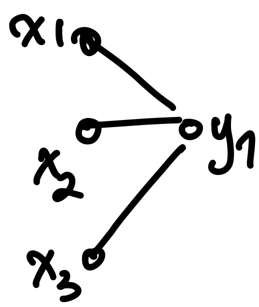
for $m,n \geq 1$

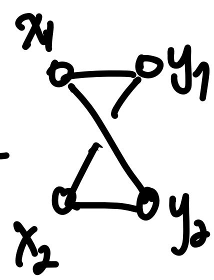
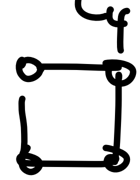
$$V = X \sqcup Y, \quad X = \{x_1, \dots, x_m\}, \quad Y = \{y_1, \dots, y_n\}$$

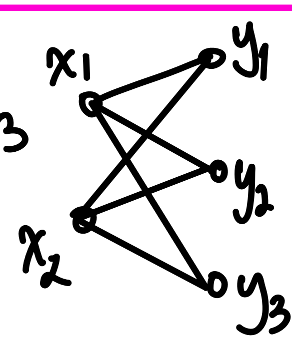
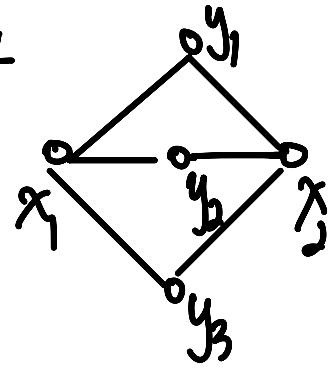
$$E = \left\{ \text{all pairs } \{x_i, y_j\} \;\middle|\; \begin{array}{l} 1 \leq i \leq m \\ 1 \leq j \leq n \end{array} \right\}$$

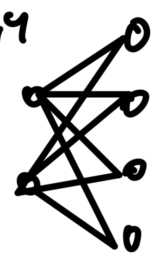
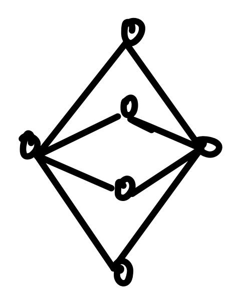
$K_{1,1}$  $\cong K_2$

$K_{1,2}$  $\cong P_2$

$K_{1,3}$  $\cong K_{3,1}$ 

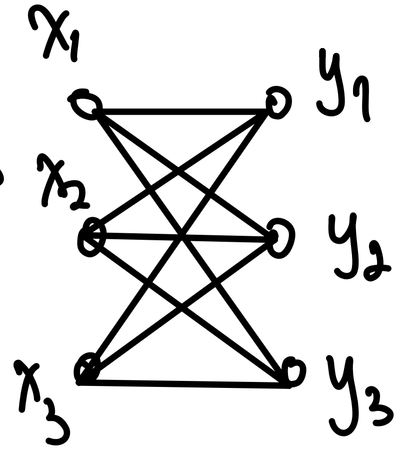
$K_{2,2}$  $\cong$ C_4 

$K_{2,3}$  $\cong$ 

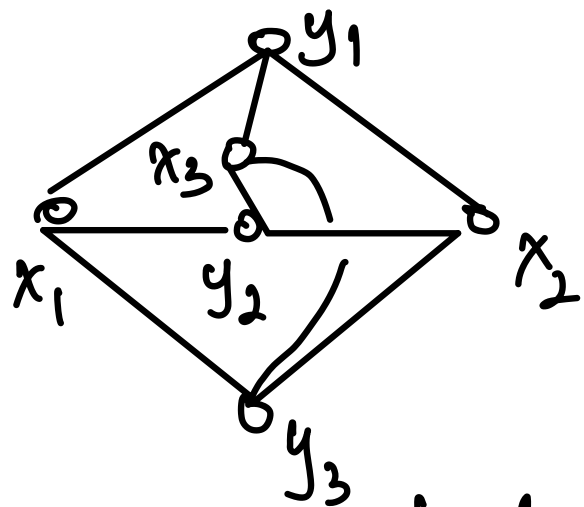
$K_{2,4}$  $\cong$ 

$$\boxed{\begin{array}{l} K_{1,n} \cong K_{n,1} \\ K_{m,n} \cong K_{m,n} \end{array}}$$

$K_{3,3}$



$\cong$



Seems impossible to draw $K_{3,3}$ in the plane $\mathbb{R}^2$ without some crossing edges; not planar, similar to K_5.

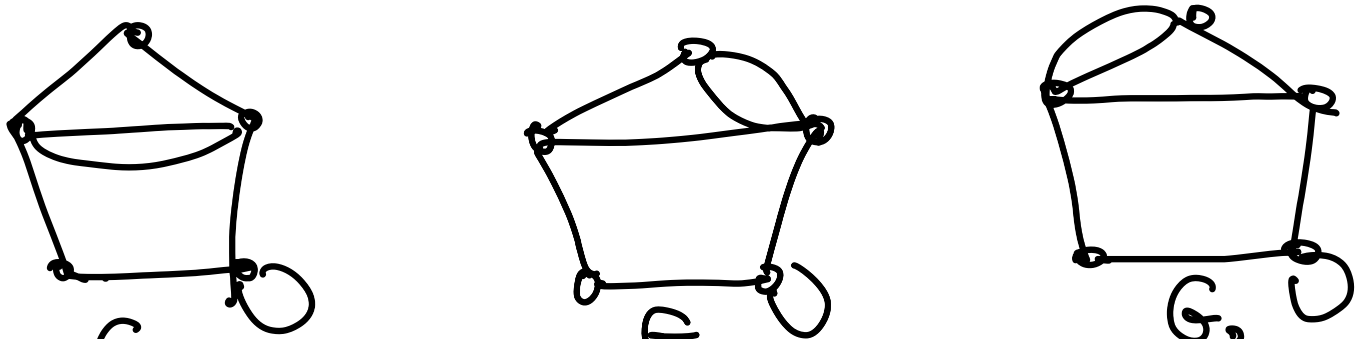
We'll return to this with more tools later.

ACTIVE LEARNING

2 min? (think-pair-share)

(Bondy-Murty Exercise 1.2.3)

Prove that no pair of these multigraphs G_1, G_2, G_3 are isomorphic, that is, $G_1 \not\cong G_2$, $G_1 \not\cong G_3$, $G_2 \not\cong G_3$.



G_1 G_2 G_3

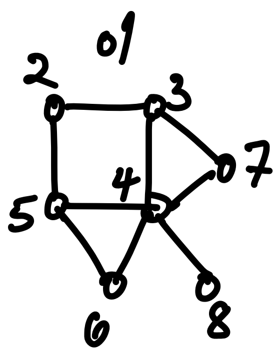
Need to think about properties isomorphisms preserve.

Degree sequences

DEF'N: In a multigraph $G = (V, E)$, the **degree** (valence) $d_G(x)$ for $x \in V$ is the number of edges $e \in E$ with $x \in e$ (x incident to e), and self-loops at x count twice!

EXAMPLE

$G =$

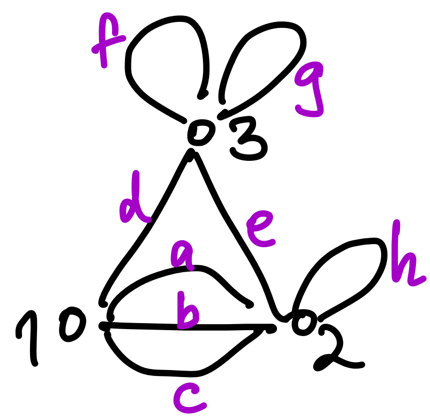


has

x	$d_G(x)$
1	0
2	2
3	3
4	5
5	3
6	2
7	2
8	1

TOTAL: 18

$G =$



has

x	$d_G(x)$
1	4
2	6
3	6

TOTAL: 16

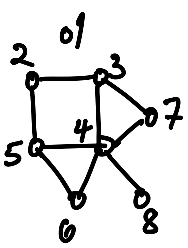
Q: What are these totals?

PROPOSITION: In any multigraph,

$$\sum_{x\in V} d_G(x) = 2\cdot|E|$$

In particular, $\sum_{x\in V} d_G(x)$ is always even, so the number of $x\in V$ having $d_G(x)$ odd is always even.

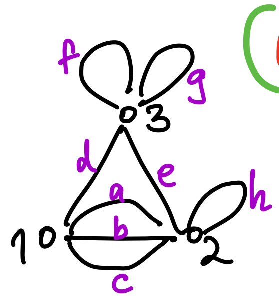
EXAMPLE 0, 2, 3, 5, 3, 2, 2, 1



4 of them are odd

even!

4, 6, 6



0 of them are odd

even!

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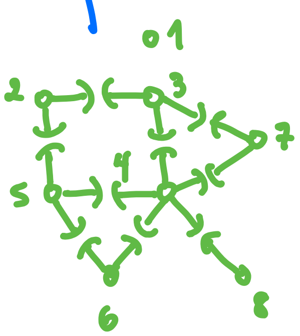
proof of PROP: Count the half-edges



$$H(G) := \{(x,e) : x\in V,\ e\in E \text{ with } x,e \text{ incident}\}$$

in 2 ways:

$$\sum_{x\in V}\deg_G(x) = |H(G)| = \sum_{e\in E} 2 = 2\cdot|E|$$

$\blacksquare$

COROLLARY:

A multigraph $G=(V,E)$ has average vertex degree

$$\frac{1}{|V|}\sum_{x\in V} d_G(x) = \frac{2|E|}{|V|}.$$

In particular, if G is d-regular, meaning $d_G(x)=d \ \forall x\in V$, then $d=\frac{2|E|}{|V|}$.

EXAMPLES

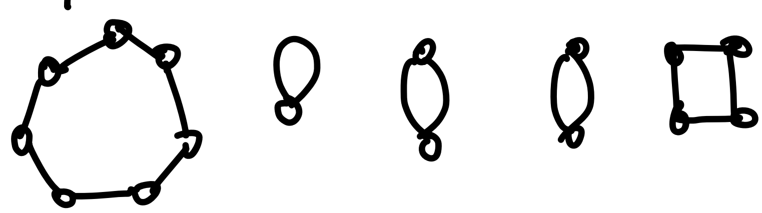
(1) 2-regular multigraphs have $2\frac{|E|}{|V|}=2$, so $|E|=|V|$.

What can they look like?

PROPOSITION: A finite 2-regular multigraph G is a disjoint union of cycles, i.e.

$$G = C_{m_1}\sqcup C_{m_2}\sqcup\dots\sqcup C_{m_\ell} \text{ for some } m_1,\dots,m_\ell\geq 1.$$

e.g. $G = C_7 \sqcup C_1 \sqcup C_2 \sqcup C_2 \sqcup C_4$



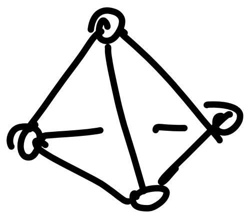
ACTIVE LEARNING

Prove this PROPOSITION.

② 3-regular graphs are sometimes called cubic graphs

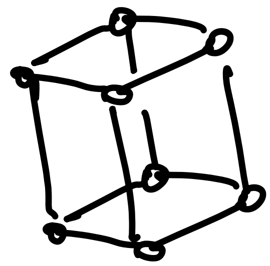
They have $\frac{2|E|}{|V|} = 3$, so $\frac{|E|}{|V|} = \frac{3}{2}$.

K_4



$|E| = 6$

$|V| = 4$ $\qquad \frac{6}{4} = \frac{3}{2}$

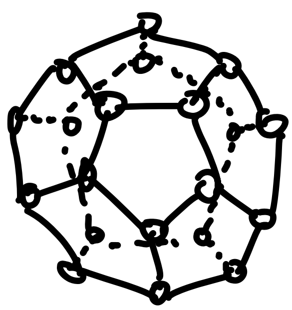


3-cube graph

$|E| = 12$

$|V| = 8$ $\qquad \frac{12}{8} = \frac{3}{2}$

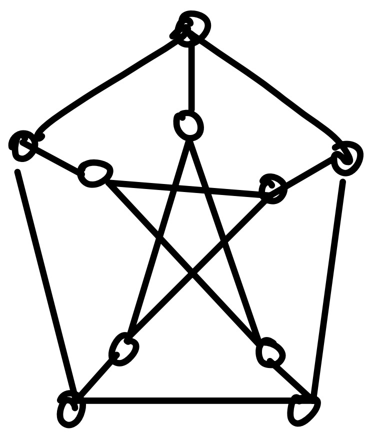
dodecahedron graph



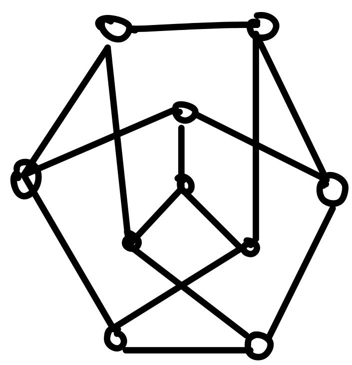
$|E| = 30$

$|V| = 20$

$\frac{30}{20} = \frac{3}{2}$



$\cong$



Petersen graph
(gives lots of counterexamples)

$|E| = 15$

$|V| = 10$ $\qquad \frac{15}{10} = \frac{3}{2}$

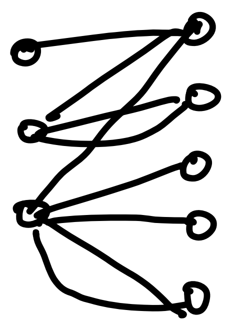
Similar to $\sum_{x \in V} d_G(v) = 2 \cdot |E|$ ...

PROPOSITION: In any bipartite multigraph $G = (V, E)$, $V = X \sqcup Y$, one has

$$\sum_{x \in X} d_G(x) = |E| = \sum_{y \in Y} d_G(y).$$

EXAMPLE

$d_G(x)$	X	Y	$d_G(y)$
1			3
3			2
5			1
			1
			1
TOTAL: 8			TOTAL: 8



$|E| = 8$

proof: Count $|E|$ in two ways:

$$\sum_{x \in X} d_G(x) = |E| = \sum_{y \in Y} d_G(y)$$

every edge $e = \{x, y\}$ has a (unique) end-vertex in X. ∎

COROLLARY: In any bipartite multigraph $G = (V, E)$, $V = X \sqcup Y$,

$$\frac{\text{average degree for } x \text{ in } X}{\text{average degree for } y \text{ in } Y} = \frac{|Y|}{|X|}$$

proof: LHS "left hand side"

$$= \frac{\frac{1}{|X|} \sum_{x \in X} d_G(x)}{\frac{1}{|Y|} \sum_{y \in Y} d_G(y)} = \frac{\frac{1}{|X|} \cdot \cancel{|E|}}{\frac{1}{|Y|} \cdot \cancel{|E|}} = \frac{|Y|}{|X|}$$ ∎

REMARK:

Compare this with the discussion in the NY Times article (Aug. 12, 2007) by Gina Kolata "The Myth, the Math, the Sex"

In recent U.S., British surveys asking people their # of lifetime heterosexual sex partners of opposite gender:

	average for women (X)	average for men (Y)
British study	6.5	12.7
U.S. study	~4	~7

Note $\underset{\#\text{women}}{\underset{\shortparallel}{|X|}} \sim \underset{\#\text{men}}{\underset{\shortparallel}{|Y|}}$ in both studies, so somebody is lying.

Amusingly, Kolata interviews renowned expert on bipartite graphs and matching theory, David Gale. His name will come up in

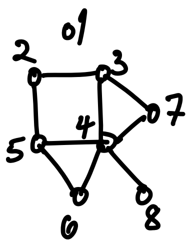
Gale-Ryser Theorem

Gale-Shapley Algorithm

Q: What can one say about the vertex degree sequences $\underline{d}(G) = (d_G(1), d_G(2), \ldots, d_G(n))$
$= (d_1, d_2, \ldots, d_n)$

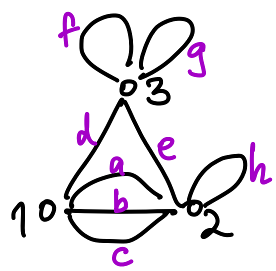
that can appear for multigraphs G?
for simple graphs G?

By relabeling the vertices $1, 2, \ldots, n$, one can assume $d_1 \geq d_2 \geq \ldots \geq d_n$



$(0,2,3,5,3,2,2,1)$

$\underline{d}(G) = (5,3,3,2,2,2,1,0)$



$(4,6,6)$

$\underline{d}(G) = (6,6,4)$

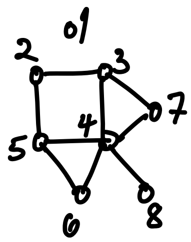
Certainly $\left\{ \begin{array}{c} d_i \in \mathbb{Z} \\ d_i \geq 0 \\ \sum_{i=1}^{n} d_i \text{ even} \end{array} \right\}$ are all necessary conditions for $\underline{d} = (d_1, \ldots, d_n) = \underline{d}(G)$ for some G

PROPOSITION: (easy) (Bondy-Murty EXER 1.5.5 on HW 1) $\underline{d} = (d_1, \ldots, d_n)$ has $\underline{d} = \underline{d}(G)$ for some multigraph G $\iff$ $\left\{ \begin{array}{l} d_i \in \mathbb{Z} \text{ and} \\ d_i \geq 0 \text{ and} \\ \sum_{i=1}^{n} d_i \text{ even} \end{array} \right.$

The question becomes much trickier for $\underline{d} = \underline{d}(G)$ of simple graphs G.

EXAMPLE

$\underline{d} = (2)$, $(2,0)$, $(3,1)$ are impossible as $\underline{d} = \underline{d}(G)$ for simple graphs G.



achieved $\underline{d}(G) = (5,3,3,2,2,2,1,0)$

ACTIVE LEARNING:

Can you achieve $\underline{d} = (5,5,4,2,2,2) = \underline{d}(G)$ for a simple graph G?

The following gives a fast algorithmic test:

PROPOSITION (Havel, Hakimi)
1955, 1962
Bondy-Murty EXER. 1.5.7(a) on HW1

$\underline{d} = (d_1 \geq d_2 \geq \dots \geq d_n) = \underline{d}(G)$ for a simple graph G

$\iff$

$\hat{\underline{d}} = (d_2 - 1, d_3 - 1, \dots, d_{d_1+1} - 1, d_{d_1+2}, d_{d_1+3}, \dots, d_n) = \underline{d}(\hat{G})$

for a simple graph $\hat{G}$.

The backward implication ($\Leftarrow$) is easy:
if $\underline{\hat{d}} = \underline{d}(\hat{G})$, then $\underline{d} = \underline{d}(G)$ where G has an extra vertex 1 connected to vertices $2, 3, \dots, d_1+1$:

$\hat{G} =$ 3−2−4, 5 $\quad \underline{\hat{d}} = (2,1,1,0) \rightsquigarrow \underline{d} = (3,3,2,2,0) \quad G =$ 1 connected to 3, 2, 4; 3−2−4, 5

One can apply this PROPOSITION repeatedly until
either all $\hat{d}_i = 0$ (so answer is YES)
or some $\hat{d}_i < 0$ (so answer is NO).

EXAMPLES

$\underline{d} = (5,3,3,2,2,2,1,0)$

$\underline{\hat{d}} = (2,2,1,1,1,1,0)$

$\hat{\hat{d}} = (1,0,1,1,1,0)$

re-order $\rightsquigarrow (1,1,1,1,0)$

$\hat{\hat{\hat{d}}} = (0,1,1,0)$

re-order $\rightsquigarrow (1,1,0,0)$

$\hat{\hat{\hat{\hat{d}}}} = (0,0,0)$

YES

$\underline{d} = (5,5,4,2,2,2)$

$\underline{\hat{d}} = (4,3,1,1,1)$

$\hat{\hat{d}} = (2,0,0,0)$

$\hat{\hat{\hat{d}}} = (-1,-1,0)$

NO

There are some other more explicit and algorithmically faster criteria:

THEOREM (Erdös-Gallai 1960) $\underline{d} = (d_1 \geq d_2 \geq \ldots \geq d_n) = \underline{d}(G)$ for a simple graph G

$$\iff \begin{cases} d_i \in \mathbb{Z} \text{ and} \\ d_i \geq 0 \text{ and} \\ \sum_{i=1}^{n} d_i \text{ even and} \\ \forall k = 1, 2, \ldots, n, \quad d_1 + d_2 + \ldots + d_k \leq k(k-1) + \sum_{i=k+1}^{n} \min(d_i, k) \end{cases}$$

EXAMPLE $\underline{d} = (5, 5, 4, 2, 2, 2)$

has at $k=1$: $d_1 = 5 \overset{?}{\leq} 1 \cdot 0 + (1+1+1+1+1)$ ✓

$k=2$: $d_1 + d_2 = 5+4 \overset{?}{\leq} 2 \cdot 1 + (2+2+2+2)$ ✓

$k=3$: $d_1 + d_2 + d_3 = 5+5+4 \overset{?}{\leq} 3 \cdot 2 + (2+2+2)$ NO.

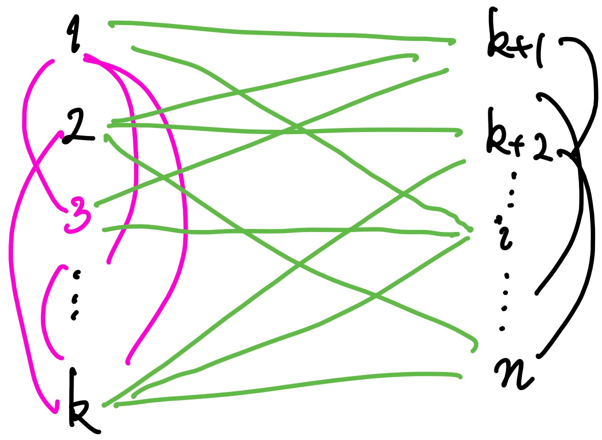
proof of ($\Rightarrow$)

[backward implication $\Leftarrow$ takes work!]

Fixing k, let's count

$$\left\{ \text{half-edges } (x, e) \begin{matrix} x \in V \\ e \in E \end{matrix} : \begin{matrix} x \text{ incident to } e = \{x, y\} \\ x \in \{1, 2, \ldots, k\} \end{matrix} \right\}$$

They are the half-edges inside pink and green edges below, ignoring the black ones.



$\leq (k-1)k$ $\qquad$ $\leq k$, $\leq d_i$

There are exactly $d_1 + d_2 + \cdots + d_k$ such half-edges (x, e), by classifying them according to $x = i$ for $i = 1, 2, \ldots, k$.

We can upper bound them by first bounding the pink ones, where $e = \{x, y\}$ has both $x, y \in \{1, 2, \ldots, k\}$, by $k(k-1)$, as there at most $k-1$ such pink half-edges emanating from each $x = 1, 2, \ldots, k$.

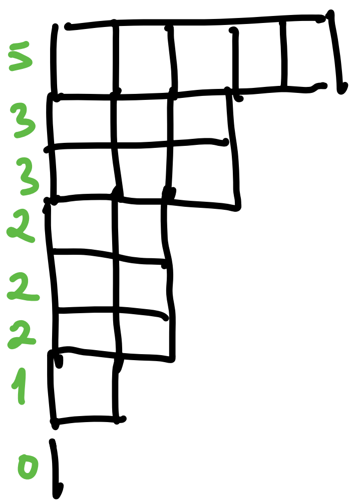
The green ones $\{x, y\}$ emanating from some $y = k+1, \ldots, n$ have size at most d_i, but also at most k since $x \in \{1, 2, \ldots, k\}$. Hence there are at most $\min\{k, d_i\}$.

Thus $d_1 + \cdots + d_k \leq k(k-1) + \sum_{i=k+1}^{n} \min\{k, d_i\}$ $\blacksquare$

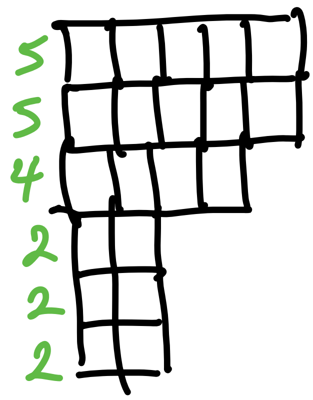
Here is another explicit, algorithmically fast criterion that first views $\underline{d}=(d_1\geq d_2\geq\dots\geq d_n)$ as a Ferrers diagram with d_i squares flush left in row i

EXAMPLE

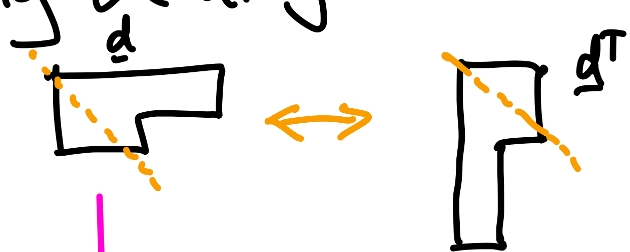
$\underline{d}=(5,3,3,2,2,2,1,0)$



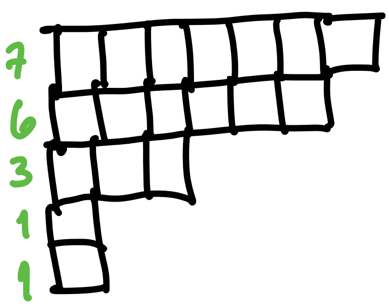
$\underline{d}=(5,5,4,2,2,2)$



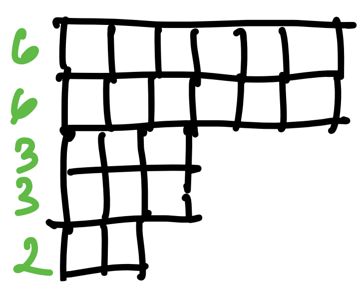
and compares them with their transpose $\underline{d}^T$ obtained by flipping the diagram across the 45° diagonal:



$\underline{d}^T=(7,6,3,1,1)$



$\underline{d}^T=(6,6,3,3,2)$



THEOREM (Ruch & Gutman 1979)

$\underline{d} = (d_1 \geq d_2 \geq \dots \geq d_n) = \underline{d}(G)$ for a simple graph G

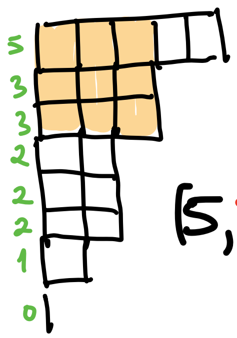
$$\iff \begin{cases} d_i \in \mathbb{Z} \text{ and} \\ d_i \geq 0 \text{ and} \\ \sum_{i=1}^{n} d_i \text{ even and} \\ (d_1+1)+\dots+(d_k+1) \leq d_1^T + d_2^T + \dots + d_k^T \\ \quad \text{for } k = 1, 2, \dots, \operatorname{rank}(\underline{d}) := \max\{i : \lambda_i \geq i\} \end{cases}$$

We'll skip the proof, but again $(\Rightarrow)$ is easier
$(\Leftarrow)$ is harder.

EXAMPLE

Both $\underline{d}$ above have $\operatorname{rank}(\underline{d}) = 3$

$\underline{d} =$



$(5,3,3,2,2,2,1,0)$

$\underline{d}^T = (7,6,3,1,1)$

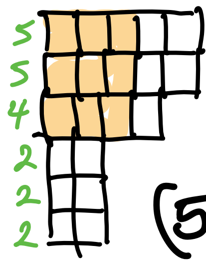
$5+1 \leq 7$ ✓

$(5+1)+(3+1) \leq 7+6$ ✓

$(5+1)+(3+1)+(3+1) \leq 7+6+3$ ✓

YES

$\underline{d} =$



$(5,5,4,2,2,2)$

$\underline{d}^T = (6,6,3,3,2)$

$5+1 \leq 6$ ✓

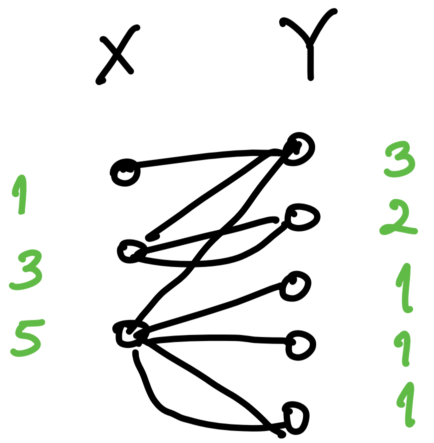
$(5+1)+(5+1) \leq 6+6$ ✓

$(5+1)+(5+1)+(4+1) \not\leq 5+5+4$

NO.

For bipartite multigraphs $G = (V, E)$, $V = X \sqcup Y$

and their bipartite degree sequences $(\underline{d}^X, \underline{d}^Y)$



$(\underline{d}^X, \underline{d}^Y)$

$= ((1,3,5), (3,2,1,1,1))$

$\rightsquigarrow$ reorder $((5,3,1), (3,2,1,1,1))$

... there is a similar, but somewhat simpler story.

PROPOSITION: (easy) $(\underline{d}^X, \underline{d}^Y)$ weakly decreasing sequences come from a bipartite multigraph

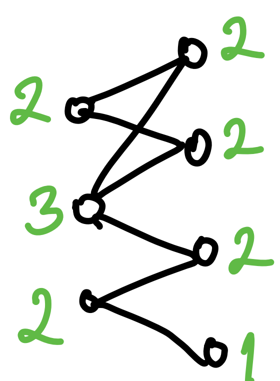
$$\Longleftrightarrow \begin{cases} d_i^X, d_i^Y \in \mathbb{Z}, \geq 0 \\ \text{and} \\ \sum_{i=1}^{m} d_i^X = \sum_{j=1}^{n} d_j^Y \end{cases}$$

THEOREM (Gale, Ryser 1963) $(\underline{d}^X, \underline{d}^Y)$ weakly decreasing sequences come from a bipartite simple graph

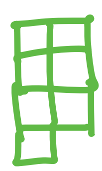
$$\Longleftrightarrow \begin{cases} d_i^X, d_i^Y \in \mathbb{Z}, \geq 0 \text{ and } \sum_{i=1}^{m} d_i^X = \sum_{j=1}^{n} d_j^Y \\ \text{and} \\ d_1^X + \dots + d_k^X \leq (d^Y)^T_1 + \dots + (d^Y)^T_k \quad \forall k = 1, \dots, m \end{cases}$$

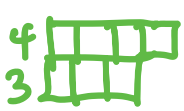
Again, the $(\Rightarrow)$ implication is easier to prove, but let's skip it. The $(\Leftarrow)$ implication is also not as hard to prove in the bipartite case.

EXAMPLES

$G =$ 

$d^X = (3,2,2)$ $\qquad$ $d^Y = (2,2,2,1)$



$\rightsquigarrow (d^Y)^T =$ 

$= (4,3)$

satisfies

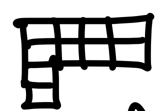
$$d_1^X = 3 \le 4 = (d^Y)^T_1$$

$$d_1^X + d_2^X = 3+2 \le 4+3 = (d^Y)^T_1 + (d^Y)^T_2$$

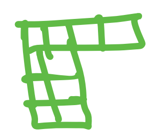
$$d_1^X + d_2^X + d_3^X = 3+2+2 \le 4+3+0 = (d^Y)^T_1 + (d^Y)^T_2 + (d^Y)^T_3$$

However

$(\underline{d}^X, \underline{d}^Y) = ((4,3,3), (4,4,1,1))$



is impossible for a simple bipartite graph,

since $(\underline{d}^Y)^T =$ 

$= (4,2,2,2)$

and

$$d_1^X = 4 \le 4 = (d^Y)^T_1$$

but

$$d_1^X + d_2^X = 4+3 \not\le 4+2 = (d^Y)^T_1 + (d^Y)^T_2$$