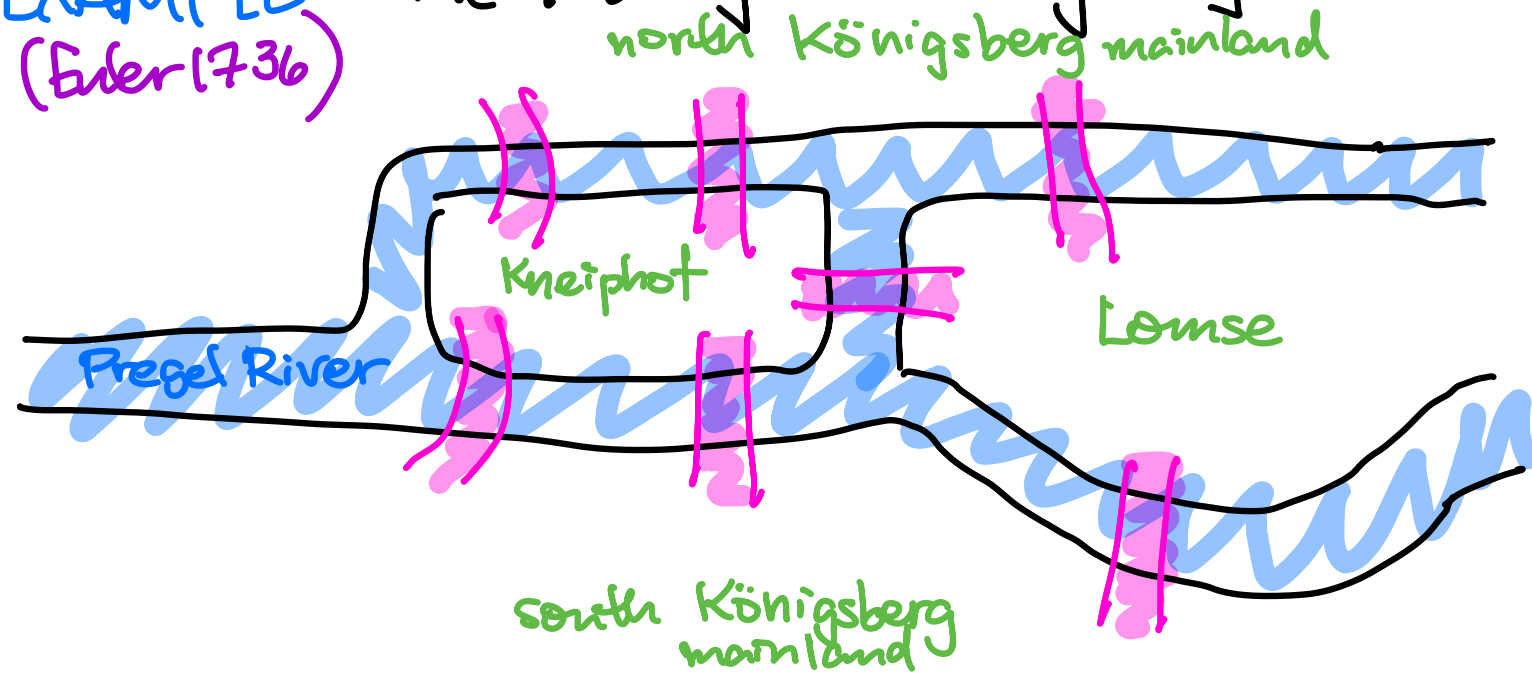
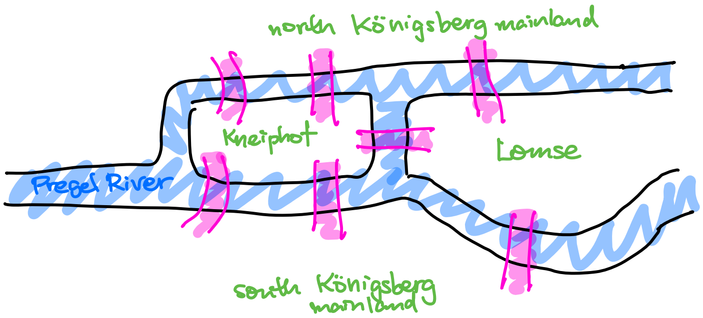
Euler tours/circuits and walks/trails
(Bondy-Murty §4.1)

EXAMPLE: The 7 bridges of Königsberg
(Euler 1736)

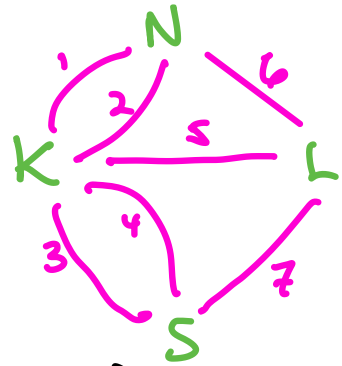


The people of Königsberg ask Leonhard Euler why they can't find a tour/circuit (= a walk with same starting and ending point) that crosses every bridge exactly once. And they can't can't even seem to find a walk/trail (i.e. with possibly different start, end points) that does this!

Euler explains it by abstracting down to a multigraph...



$\rightsquigarrow$

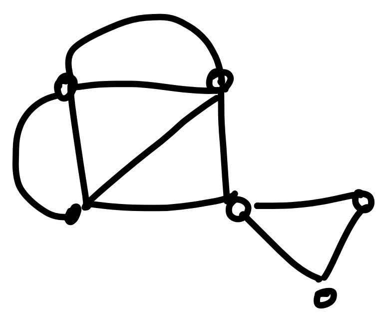


$$G = (V, E) \qquad V = \{N, S, K, L\} \qquad E = \{1,2,3,4,5,6,7\}$$

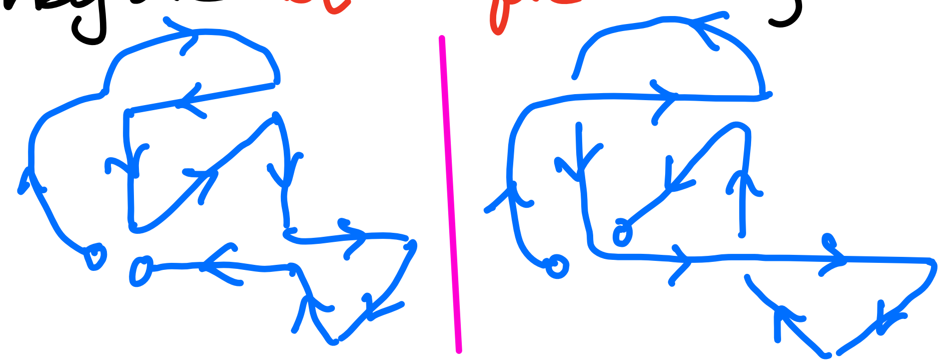
DEF'N: In a multigraph $G=(V,E)$, an **Euler walk/trail** is a walk from vertex to vertex along edges, using each edge exactly once. It's called an **Euler tour/circuit** if it starts and ends at same vertex.

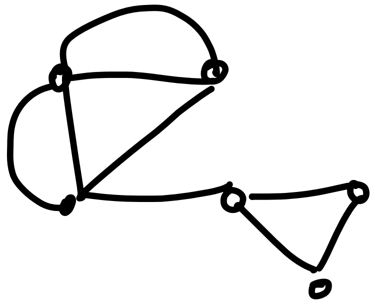
Q: Which graphs have them?
Are they **unique** in any sense?
(If not, can we **count them**?)

EXAMPLES

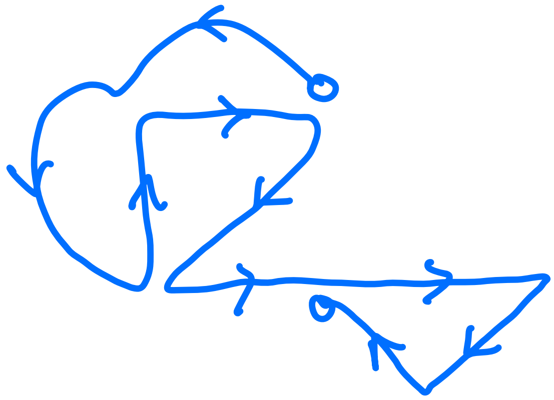
① 

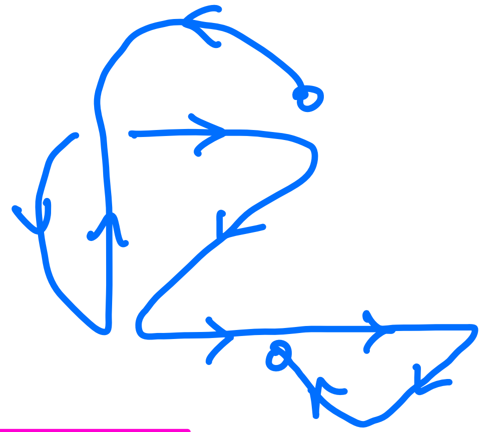
Yes, it has **Euler tours**.
They are **not unique** in any sense:



(2) 

No Euler tours, but yes, some Euler trails. Also not unique.





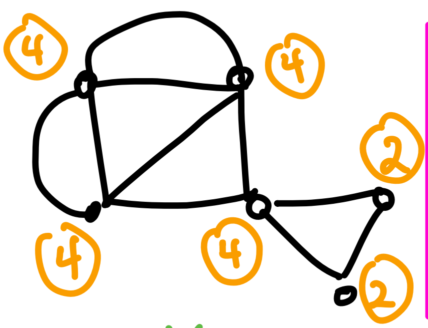
THEOREM (Euler): Let $G=(V,E)$ be a multigraph with no isolated vertices. Then

G has an Euler circuit

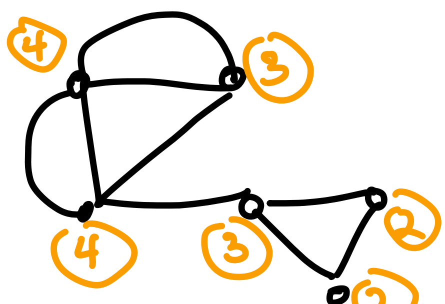
$\Longleftrightarrow$
- (a) G is connected, i.e. $\forall x,y\in V$ $\exists$ at least one path from x to y along edges of G

  AND
- (b) every vertex $x\in V$ has $\deg_G(x)$ even.

EXAMPLES

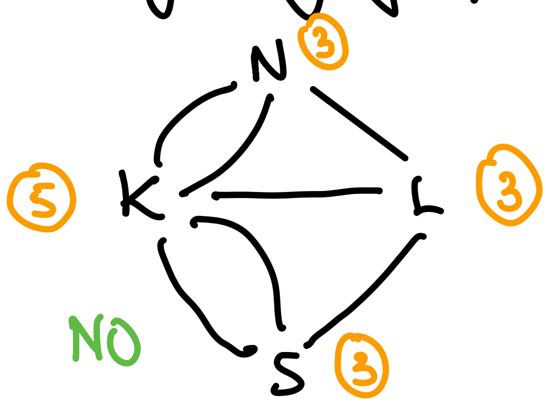


YES



NO

Königsberg graph

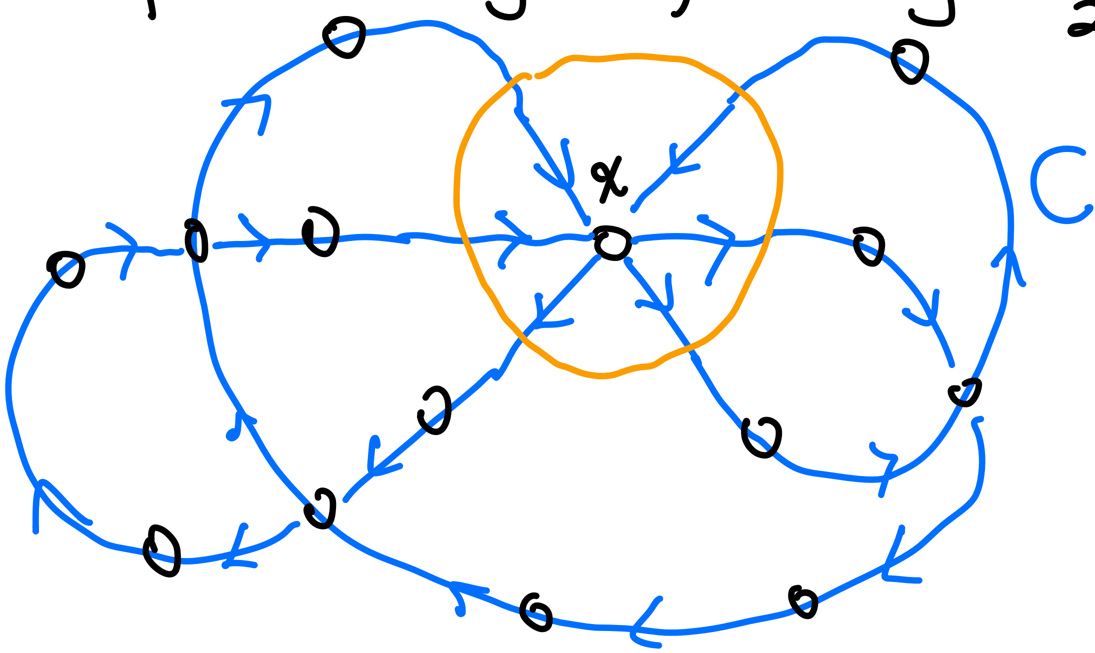


NO

proof of THEOREM:

($\Rightarrow$): Let C be a (directed) Euler circuit through G. Then $\forall x, y \in V$, $\exists$ a path from x to y inside C because both x, y touch some edge(s), and C uses every edge. So G is connected, proving (a).

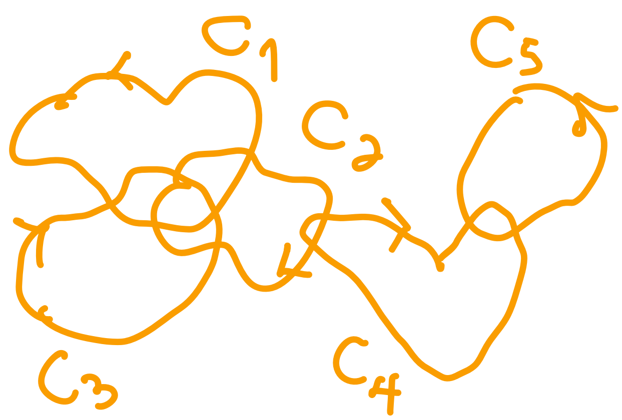
Also, for all $x \in V$, the edges e incident to x are paired off entering & exiting as C passes through x, exactly $\frac{\deg_G(x)}{2}$ times:



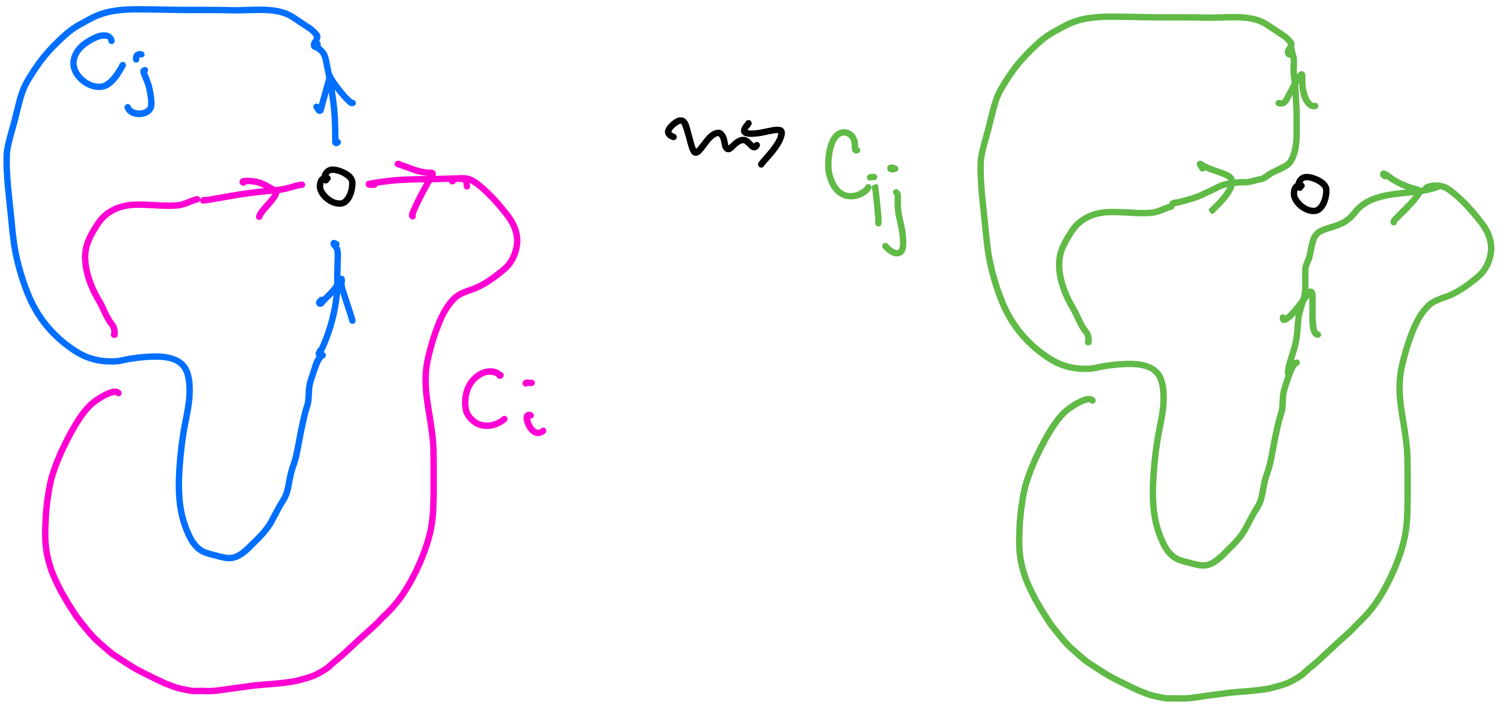
($\Leftarrow$): Assuming (a), (b) hold, here is an algorithm to produce an Euler circuit C.

Start at any vertex $x_0 \in V$. Since x_0 is not isolated, one can move along an incident edge e, and erase $e \in E$, then repeat this until you get stuck at some isolated vertex x, all of whose incident edges were erased. We claim that necessarily $x = x_0$ since every $y \neq x_0$ has even degree maintained as you enter and exit it:

Thus one has created a circuit C_1 of edges. If $C_1 = E$, we are done. Else repeat this with $G_1 = (V, E \setminus C_1)$, creating a circuit C_2. Eventually one exhausts $E = C_1 \sqcup C_2 \sqcup \cdots \sqcup C_t$.



Whenever C_i, C_j share a vertex, they can be sewn together into a single circuit $C_{ij} = C_i \cup C_j$:



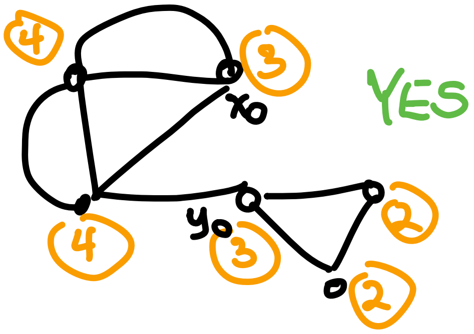
And since G is connected, one can eventually sew all of the C_i together into one Euler circuit. ▨

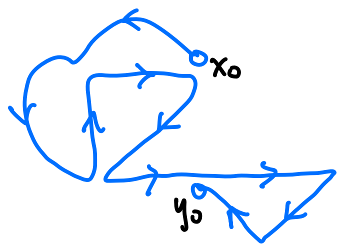
What about Euler trails?

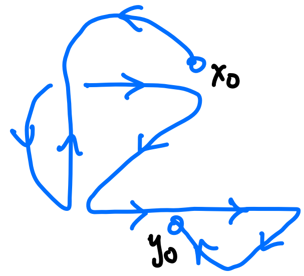
COROLLARY: Let $G=(V,E)$ be a multigraph with no isolated vertices. Then G has an Euler trail (but no Euler circuit)

$\Leftrightarrow$ (a) G is connected
AND
(b) every vertex $x \in V$ has $\deg_G(x)$ even except for two of them x_0, y_0, which will be the start and end of every Euler trail

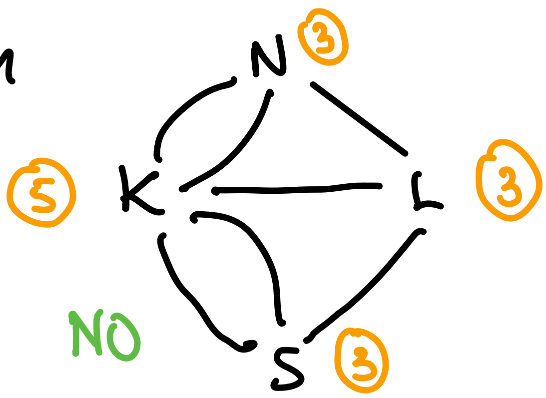
EXAMPLES





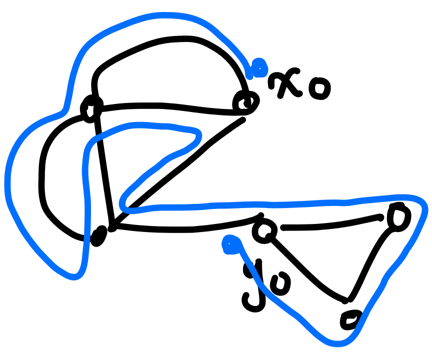


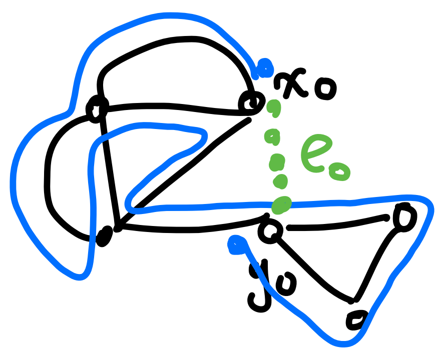
Königsberg graph



proof of COROLLARY:

($\Rightarrow$) An Euler trail $T \subset E$ again connects all the vertices in G, since none are isolated. And adding an extra edge $e_0 = \{x_0, y_0\}$ to E gives an Euler circuit $C = T \cup \{e_0\}$ in $G \cup \{e_0\} := (V, E \cup \{e_0\})$ that proves (b).





($\Leftarrow$) Conversely, if $G=(V,E)$ satisfies (a),(b), then $G \cup \{e_0\} = (V, E \sqcup \{e_0\})$ will have an Euler circuit C, and then $T := C - \{e_0\}$ will be an Euler trail in G. $\blacksquare$

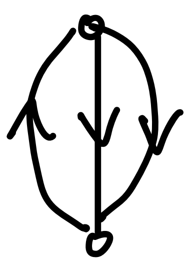
Directed Euler circuits and deBruijn sequences

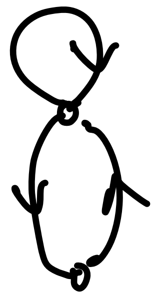
DEF'N: A directed multigraph $D=(V,A)$ (or digraph)

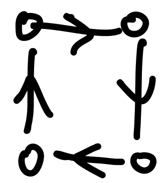
vertices = V, arcs = A

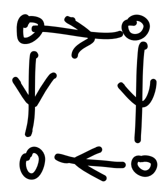
has each arc $a \in A$ an ordered pair $a=(x,y)$ thought of as a directed edge $x \xrightarrow{a} y$

EXAMPLES

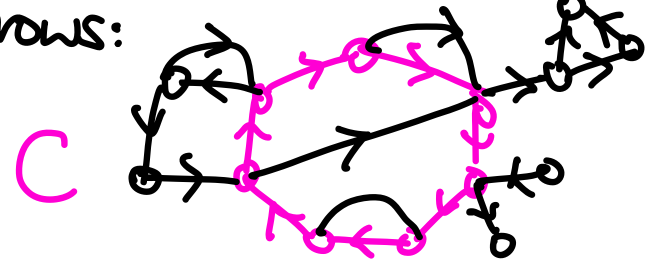
D_1 

$D_2 =$ 

$D_3 =$ 

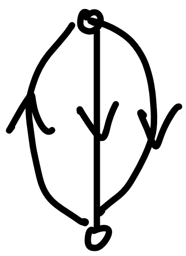
$D_4 =$ 

DEF'N: A directed circuit C in a digraph is a set of arcs $x_0 \to x_1 \to x_2 \to \cdots \to x_{\ell-1} \to x_\ell = x_0$, that is, forming a circuit that respects the arrows:

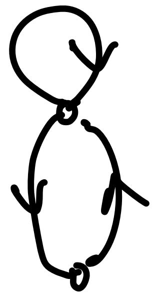


A **directed Euler tour** in a digraph $D=(V,A)$ is a directed circuit C using every arc $a\in A$ **exactly once**.

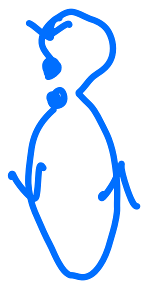
EXAMPLES

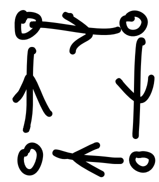
D_1 

has none

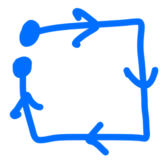
$D_2 =$ 

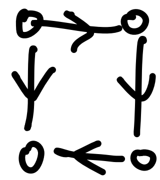
has one,

e.g. 

$D_3 =$ 

has one, namely $A=C$

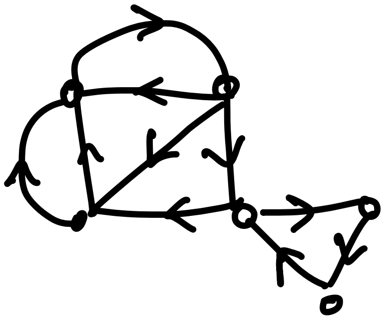


$D_4 =$ 

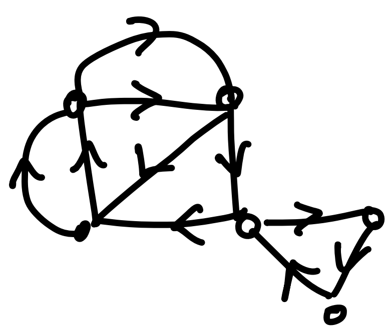
has none

ACTIVE LEARNING

Which of these digraphs has a directed Euler tour?



D_1



D_2

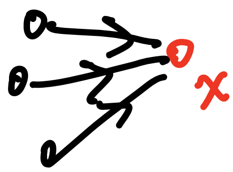
THEOREM: Let $D=(V,A)$ be a digraph with no isolated vertices. Then

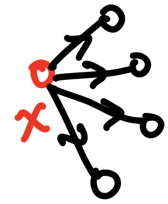
D has a directed Euler tour $\Longleftrightarrow$
(a) the underlying (undirected) graph $G=(V,E)$ is connected, where $E = \{\{x,y\} : (x,y) \in A\}$

AND

(b) $\forall x \in V$ one has $\mathrm{indeg}_D(x) = \mathrm{outdeg}_D(x)$

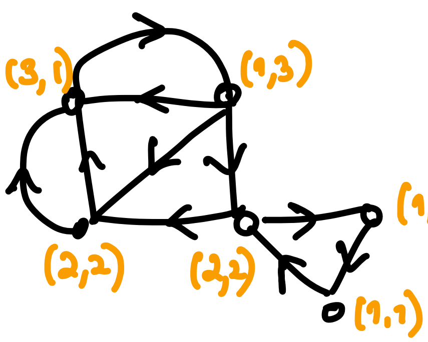
$\mathrm{indeg}_D(x) \overset{\text{DEF'N}}{=} |\{\text{arcs } (y,x) \text{ in } A\}|$, $\mathrm{outdeg}_D(x) \overset{\text{DEF'N}}{=} |\{\text{arcs } (x,y) \text{ in } A\}|$





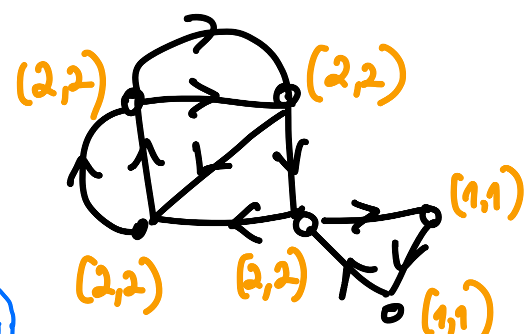
EXAMPLES with (indeg(x), outdeg(x)) labeled

D_1

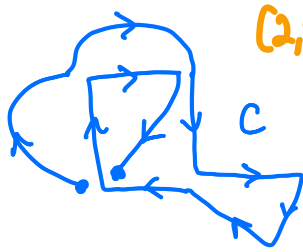


NO

D_2



YES



proof: Essentially the same proof as for Euler's Theorem on undirected Euler tours! ∎

REMARK: Later we'll learn how to count directed Euler tours.

De Bruijn sequences

- it's not apparent that they relate to directed Euler tours, but we'll see how they do!

DEF'N: A de Bruijn sequence
- on k letters $\{0,1,2,\dots,k-1\}$ ("k-ary")
- of order n

is a circularly read sequence of k^n letters $(a_1, a_2, \dots, a_{k^n})$ in which each possible word of length n appears exactly once as a consecutive subword.

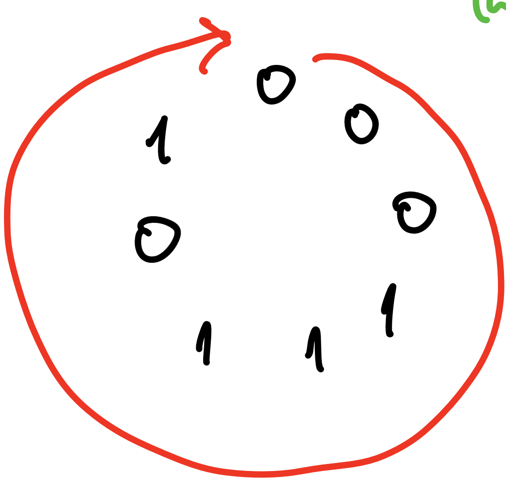
EXAMPLES

$k=2$ so letters $\{0,1\}$ "binary"

$n=3$ $(a_1, a_2, \dots, a_{2^3})$

$= (0,0,0,1,1,1,0,1)$

wrap around



consecutive subwords (a_i, a_{i+1}, a_{i+2}):

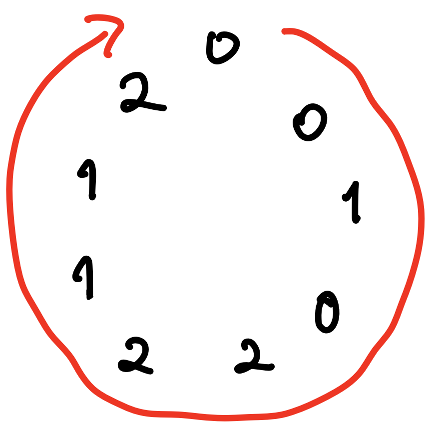
000
001
011
111
110
101
010
100

$k=3$, letters $\{0,1,2\}$ "ternary"

$n=2$

$(a_1, a_2, \dots, a_{3^2})$

$= (0,0,1,0,2,2,1,1,2)$



consecutive subwords (a_i, a_{i+1}):

00
01
10
02
22
21
11
12
20

EXAMPLE with $k=2$ and $n=5$ stolen from Wikipedia:

1	1	1	0	0	0	0	0	1	0	0	0	1	1	0	0
AS	10C	4C	7D	5H	4D	JD	KH	7C	5D	8H	QD	AC	6S	7H	AH

1	0	1	0	0	1	1	1	0	1	0	1	1	0	1	1
4S	6H	8S	9H	8D	8C	KS	10S	3H	6C	4H	JC	3C	QH	3S	9C

The mapping to 32 cards from a deck, with $0 \to$ red cards, $1 \to$ black cards lets one do a trick of Persi Diaconis:

He tosses the 32 card deck, prepared in this circular order (bound with a rubber band) into the audience, asks a few people to do a few usual cuts (break deck in 2, put bottom half on top), then asks next 5 people to take the next card off the top. He asks those among the 5 holding black cards to stand up, and then guesses all 5 peoples' cards.

Q: Do deBruijn sequences on k letters of order n exist for all k and n?

If so, how to construct them?

THEOREM (N. de Bruijn 1946) If we define the de Bruijn digraph

$$D_{k,n} = (V, A)$$

where $V = \{0,1,2,\ldots,k-1\}^{n-1} = \left\{ \begin{array}{l} \text{all sequences } (b_1,b_2,\ldots,b_{n-1}) \text{ (length } n-1\text{)} \\ \text{with } b_i \in \{0,1,2,\ldots,k-1\} \end{array} \right\}$

$A = \left\{ \begin{array}{l} \text{arcs } (b_1,b_2,\ldots,b_{n-1}) \longrightarrow (b_2,\ldots,b_{n-1},b_n) \\ \text{for all length } n \text{ sequences } (b_1,b_2,\ldots,b_{n-1},b_n) \in \{0,1,\ldots,k-1\}^n \end{array} \right\}$

then (a) $D_{k,n}$ is connected and has

$$\mathrm{indeg}_{D_{k,n}}(x) = \mathrm{outdeg}_{D_{k,n}}(x) = k \quad \forall x \in V,$$

so it has directed Euler tours, and

(b) $\left\{ \begin{array}{l} \text{directed} \\ \text{Euler tours} \\ C \text{ in } D_{k,n} \end{array} \right\}$

are in bijection with

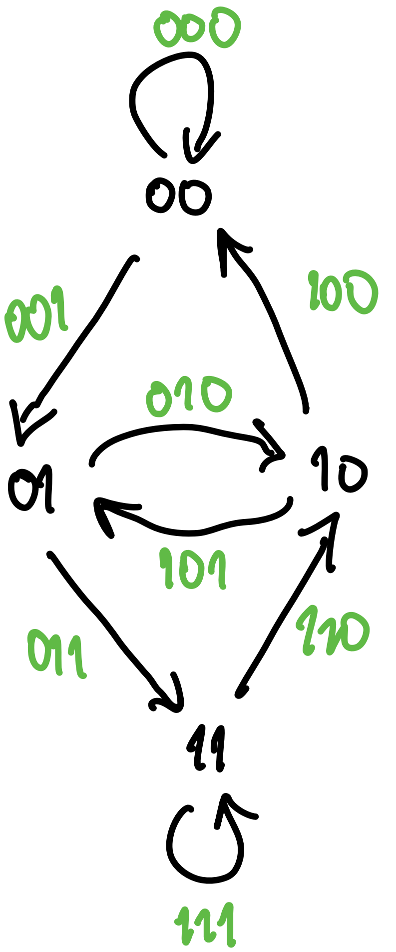
$\left\{ \begin{array}{c} k\text{-ary} \\ \text{de Bruijn sequences} \\ \text{of order } n \end{array} \right\}.$

EXAMPLES

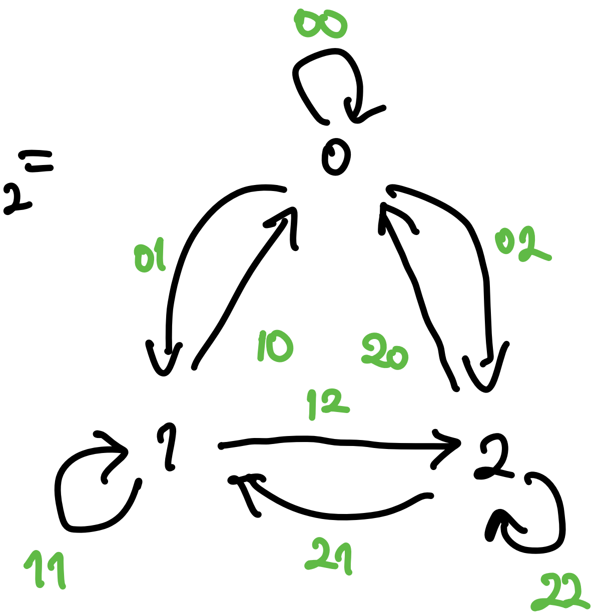
$k=2$

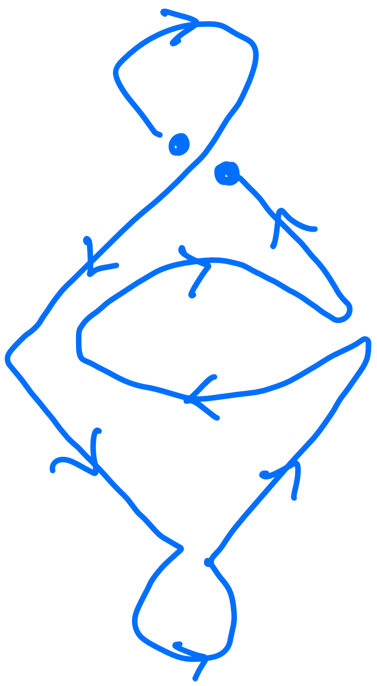
$n=3$

$D_{2,3} =$



$D_{3,2} =$

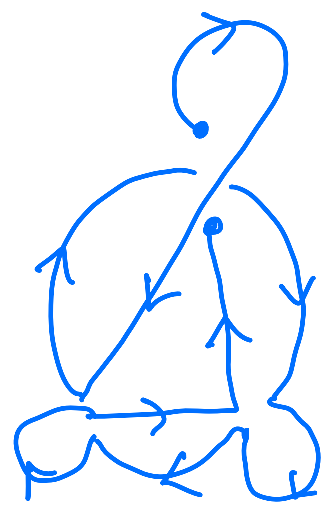




000
001
011
111
110
101
010
100

00011101

from before



00
01
10
02
22
21
11
12
20

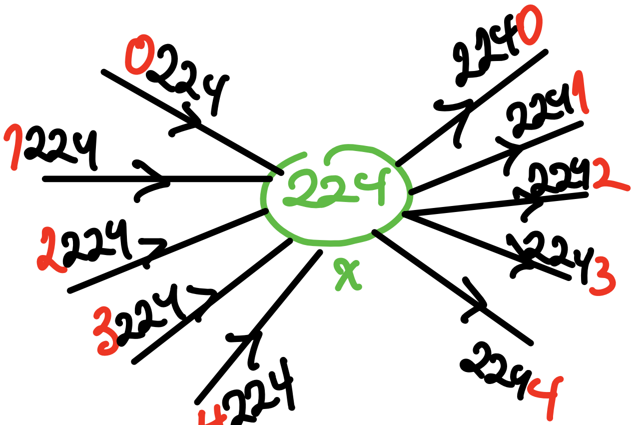
001022112

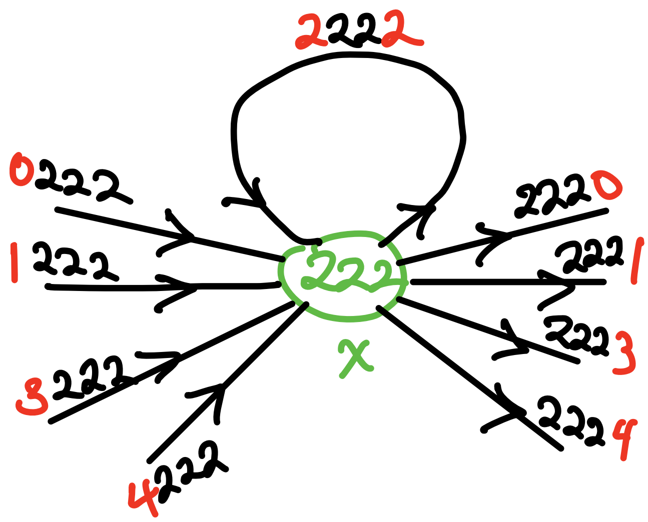
from before

proof of THEOREM:
(by EXAMPLE)

In $D_{5,4}$, one has these 2 kinds of vertices x:

$k=5$ 5-ary; order $n=4$





$\text{indeg}(x) = \text{outdeg}(x) = k$ in both cases.

And $D_{k,n}$ is connected because one can always get from x to y in $n-1$ steps, e.g. in $D_{3,6}$ ($k=3$, $n=6$)

$$\underset{x}{\underset{\parallel}{22102}} \xrightarrow{221020} 21020 \xrightarrow{210201} 10201 \xrightarrow{102012} 02012 \xrightarrow{020121} 20121 \xrightarrow{201210} \underset{y}{\underset{\parallel}{01210}}$$

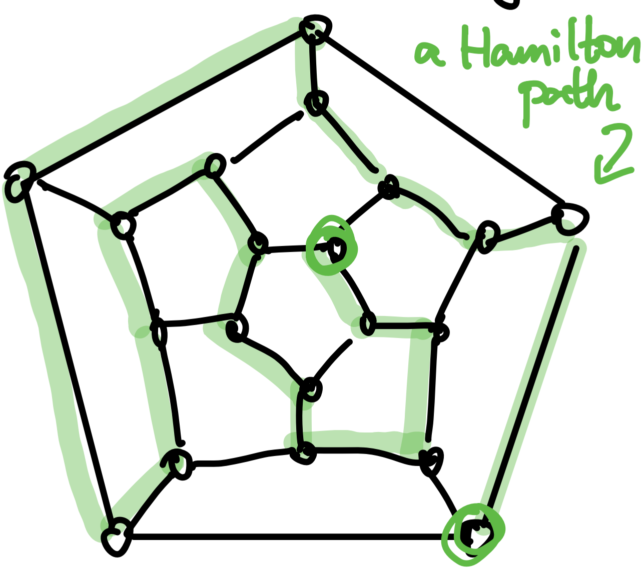
∎

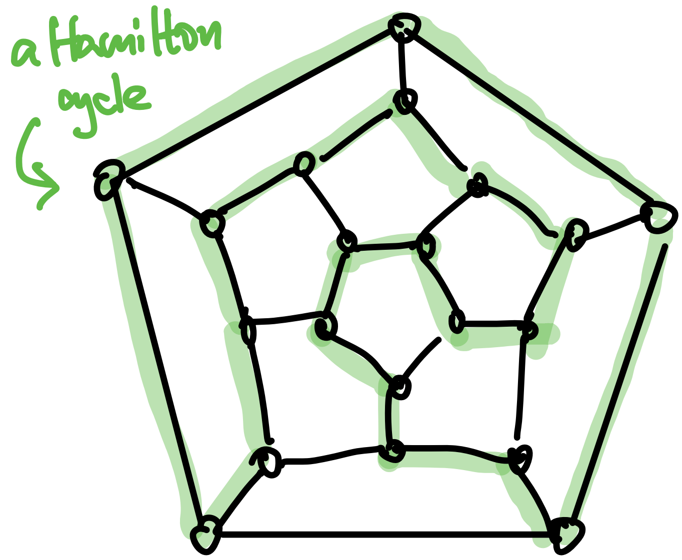
Hamilton cycles and paths

Sound very similar to Euler tours and walks, but behave surprisingly differently.

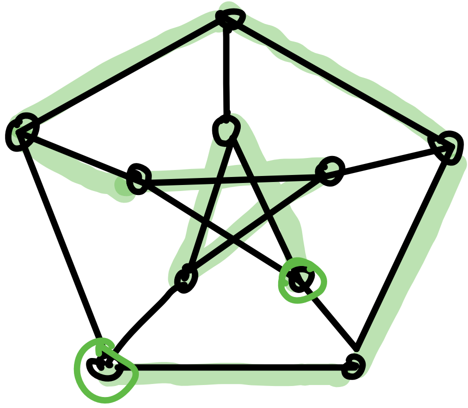
DEF'N: In a (simple) graph $G=(V,E)$, a Hamilton path is a walk from vertex to vertex along edges e in E that visits every vertex $x \in V$ exactly once (but not necessarily every edge).
If one can complete it by one last step along an edge to start and end at the same vertex, it's called a Hamilton cycle, and G is called Hamiltonian in this case.

EXAMPLE: W.R. Hamilton (1857) introduced his "Icosian game", asking one to find a Hamiltonian cycle in the dodecahedron graph:

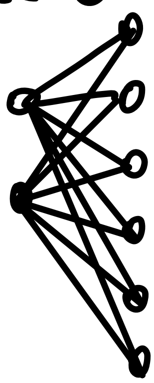
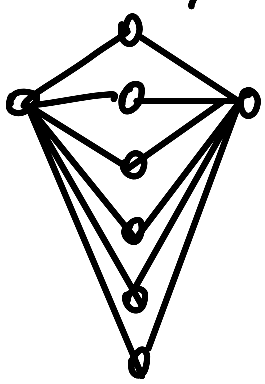




EXAMPLE: The Petersen graph has a Hamilton path, but has no Hamilton cycle (although this is not so easy to prove?)



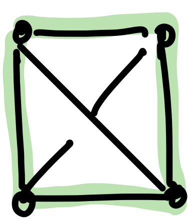
EXAMPLE: Even for simple graphs, existence of Hamilton paths/cycles is almost completely unrelated to existence of Euler trails/circuits.

e.g. $K_{2,6} =$  $\cong$ 

has an Euler circuit (why?)

but not even a Hamilton path. (why?)

Meanwhile

$K_4 =$ 

has a Hamilton cycle,

but not even an Euler walk. (why?)

Q: How can we decide whether $G=(V,E)$ is Hamiltonian?

It's not so easy.

There are some sufficient conditions that sometimes help, based on having large vertex degrees $d_G(x)$...

PROPOSITION: (Bondy-Chvátal 1976) If a simple graph $G=(V,E)$ has $\{x,y\} \notin E$ but $d_G(x)+d_G(y) \geq |V|$,

then G is Hamiltonian $\Leftrightarrow$ $G \cup \{x,y\}$ is Hamiltonian

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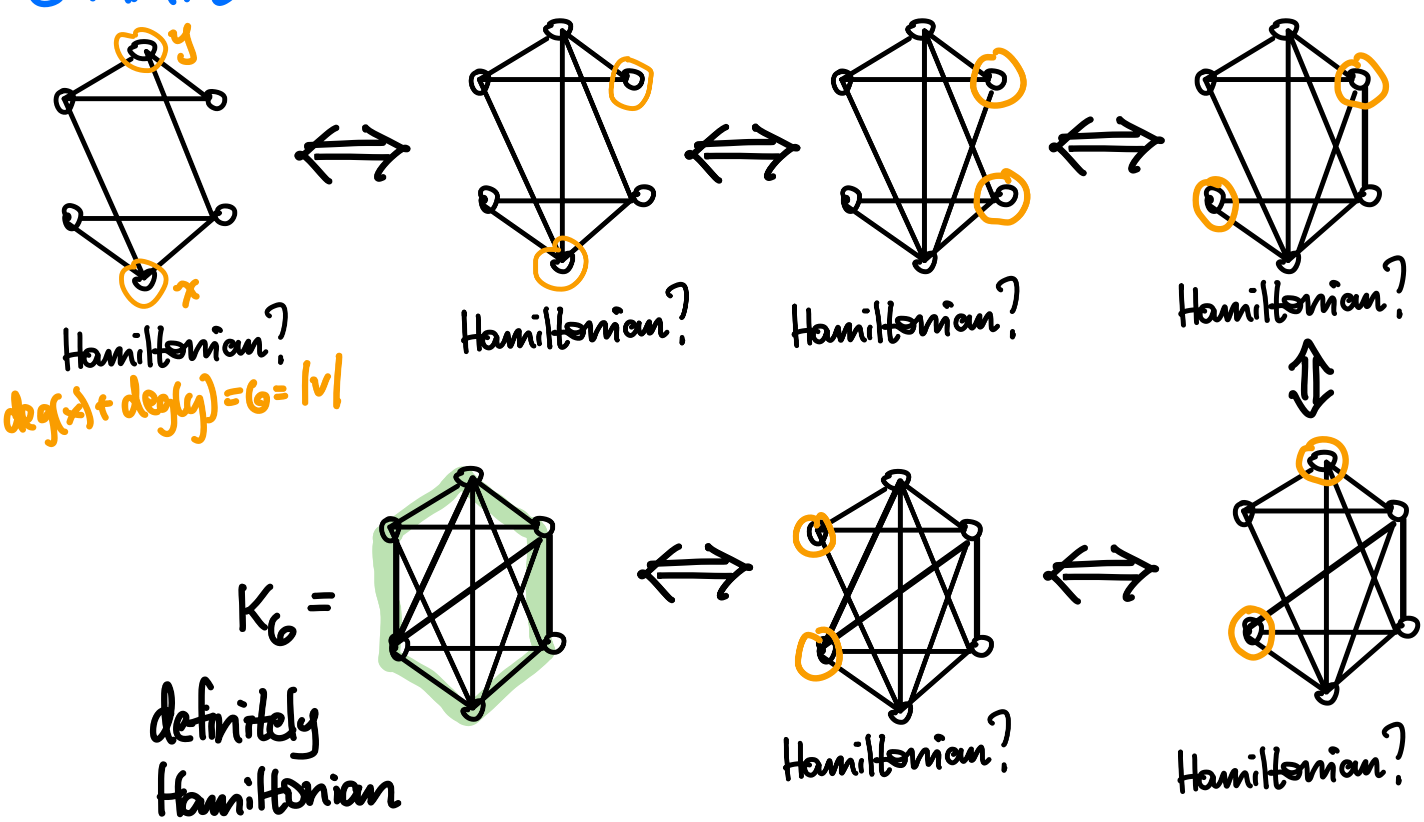
EXAMPLE



COROLLARY: (Dirac 1952) If $G=(V,E)$ has $\deg(x) \geq \frac{|V|}{2}$ $\forall x \in V$, then G is Hamiltonian.

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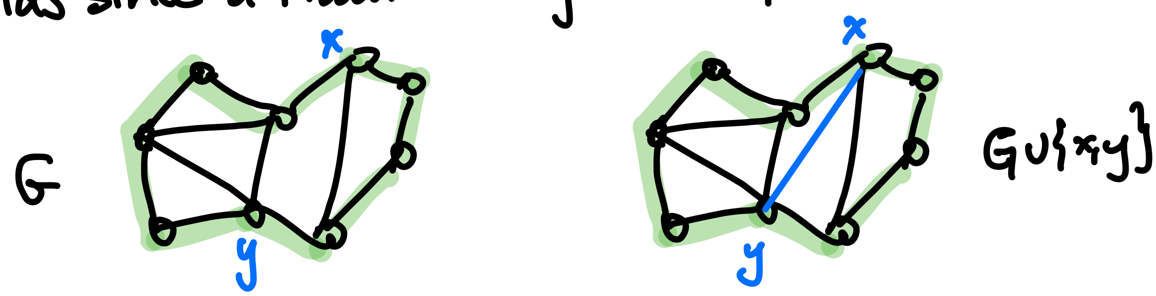
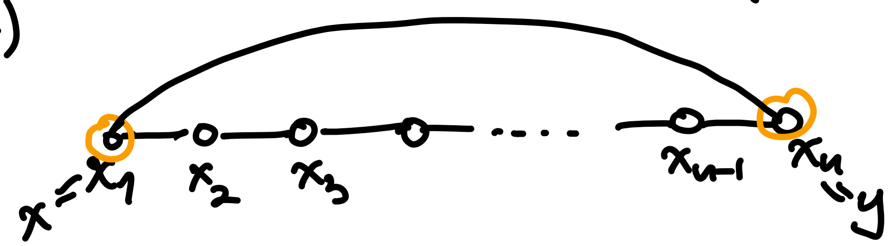
proof of PROPOSITION:

The forward implication (G Hamiltonian $\Rightarrow G \cup \{x,y\}$ Hamiltonian) holds since a Hamilton cycle in G persists in $G \cup \{x,y\}$:



For the backward implication, assume $G \cup \{x,y\}$ is Hamiltonian, and there is a Hamilton cycle C that uses $\{x,y\}$ (else it's already in G and we're done), labeled like this (so $n = |V|$):

$\overset{\shortparallel}{(V,E)}$

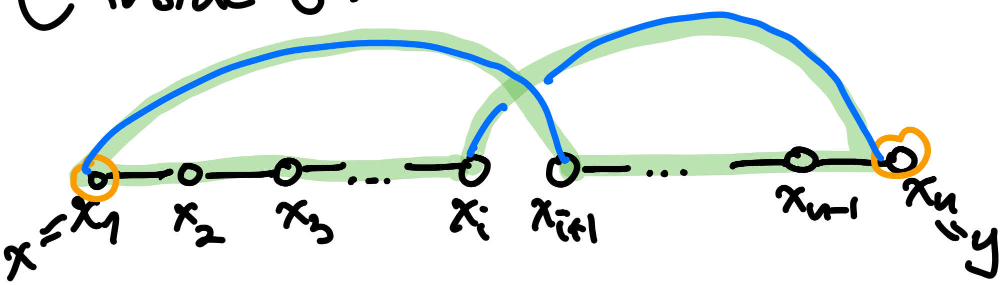


Consider the indices i in these sets:

$$X \underset{\text{DEF'N}}{=} \{ i : \{x, x_{i+1}\} \in E \} \subset \{1, 2, \ldots, n-2\}, \ |X| = d_G(x)$$

$$Y \underset{\text{DEF'N}}{=} \{ i : \{x_i, y\} \in E \} \subset \{2, 3, \ldots, n-1\}, \ |Y| = d_G(y)$$

If $X \cap Y \neq \phi$, then we're done since any $i \in X \cap Y$ gives us this construction of a Hamilton cycle C' inside G:

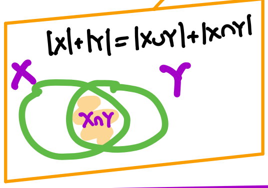


But we claim that cardinalities will indeed force $X \cap Y \neq \phi$ i.e. $|X \cap Y| \geq 1$. This is because

$$X \cup Y \subset \{1, 2, \ldots, n-1\} \quad \text{so} \quad |X \cup Y| \leq n-1,$$

but
$$|X \cap Y| = |X| + |Y| - |X \cup Y|$$

$$= \underbrace{d_G(x) + d_G(y)}_{\geq |V| = n} - \underbrace{|X \cup Y|}_{\leq n-1} \geq n - (n-1) = 1.$$



$\blacksquare$

DEF'N: Say a simple graph $G = (V, E)$ is **Bondy-Chvátal closed** if $\forall x, y \in V$ with $d_G(x) + d_G(y) \geq |V|$ one already has $\{x, y\} \in E$.

Say $\overline{G}$ is a **Bondy-Chvátal closure** of G if $\overline{G}$ is B-C closed and $\exists$ graphs on vertex set V

$$G = G_0 \subset G_1 \subset \cdots \subset G_{t-1} \subset G_t = \overline{G}$$

with $G_{i+1} = G_i \cup \{x_i, y_i\}$ having $d_{G_i}(x_i) + d_{G_i}(y_i) \geq |V|$.

PROPOSITION: The B-C closure of a graph G is unique.

proof: Consider B-C closures $\bar{G} \neq \bar{H}$ of G

$$G = G_0 \overset{e_1}{\subset} G_1 \overset{e_2}{\subset} G_2 \subset \dots \overset{e_r}{\subset} G_r = \bar{G}$$

$$G = H_0 \underset{f_1}{\subset} H_1 \underset{f_2}{\subset} H_2 \subset \dots \underset{f_s}{\subset} H_s = \bar{H}$$

with $G_{i+1} = G_i \cup \{e_{i+1}\}$ and $r+s$ minimal.
$H_{i+1} = H_i \cup \{f_{i+1}\}$

Since f_1 could be added to $G = H_0$, it could also be added to $G_1, G_2, \dots$ and hence f_1 must be an edge of $\bar{G}$, which is B-C closed. This means $f_1 = e_i$ for some i, and we can create a shorter picture:

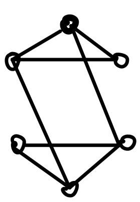
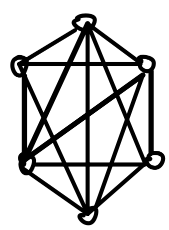
omit the step adding e_i

$$G \cup \{f_1\} = G_0 \cup \{f_1\} \overset{e_1}{\subset} G_1 \cup \{f_1\} \overset{e_2}{\subset} \dots \, \widehat{G_i \cup \{f_1\}} \dots \overset{e_r}{\subset} G_r \cup \{f_1\} = \bar{G}$$

$$G \cup \{f_1\} = H_1 \underset{f_2}{\subset} H_2 \underset{f_3}{\subset} \dots \underset{f_s}{\subset} H_s = \bar{H}.$$

This contradicts $r+s$ being minimal. ▨

COROLLARY: A simple graph G is Hamiltonian $\Longleftrightarrow$ its B-C closure $\bar{G}$ is Hamiltonian.

e.g. $G =$  had B-C closure $\bar{G} = K_6$  which is Hamiltonian

But this is not really effective in deciding whether a simple graph G is Hamiltonian.

There are plenty of interesting graphs G whose B-C closure $\bar{G} = G$ itself, but G is (non-obviously) Hamiltonian

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EXAMPLE

n-cube graphs for $n \geq 2$

$$Q_n = (V, E)$$

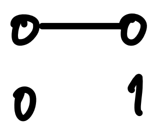
$V = \{0,1\}^n$ = all binary strings $\underline{b} = (b_1, \dots, b_n)$ of length n

$E = \{\{\underline{b}, \underline{b}'\} : \text{if } \underline{b}, \underline{b}' \text{ differ in exactly one position}\}$

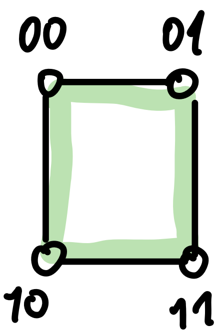
$\deg_{Q_n}(x) = n$

$|V| = 2^n$

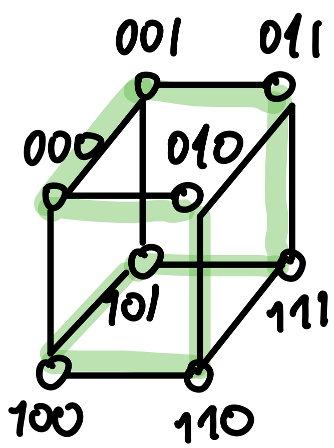
Q_1



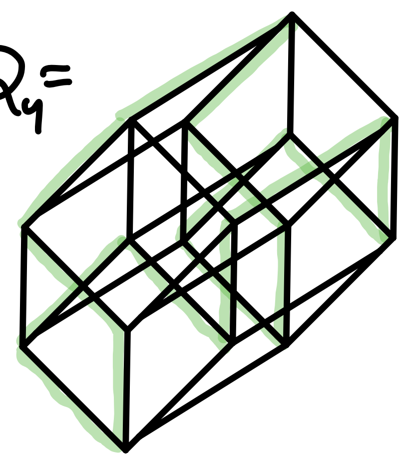
$Q_2 =$



$Q_3 =$



$Q_4 =$



Q_n is Hamiltonian for $n \geq 2$
but Q_n = its own B-C closure for $n \geq 3$.