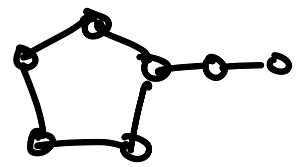
Matching theory
(Bondy & Murty, Chapter 5 and Schrijver Chapter 3)

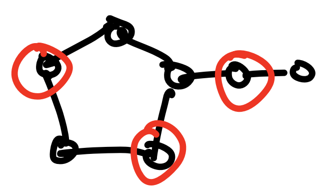
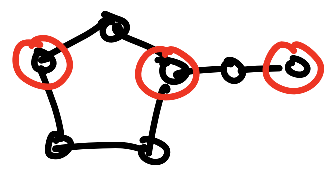
We'll learn how to relate these seemingly unrelated optima (max/mins), and compute some of them quickly.

DEFINITION: For an (undirected) simple graph $G=(V,E)$

e.g. $G=$ 

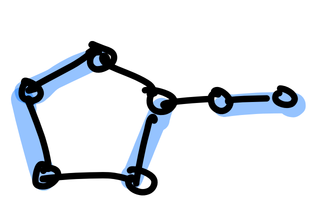
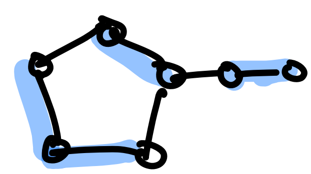
$\alpha(G) := \max\{|C| : C \subset V$ an independent set (stable)$\}$

i.e. no edges $e=\{x,y\} \in E$ have both endpoints $x,y \in V$

$=3$  or 

$\rho(G) := \min\{|F| : F \subset E$ an edge cover$\}$

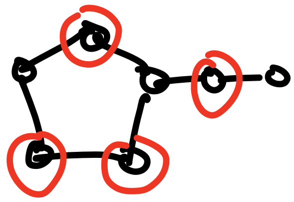
i.e. every $x \in V$ is incident to at least one edge in F

$=4$  or 

$\tau(G) := \min\{ |W| : W \subset V$ a vertex cover $\}$

i.e. every $e \in E$ is incident to at least one vertex in W

$= 4$



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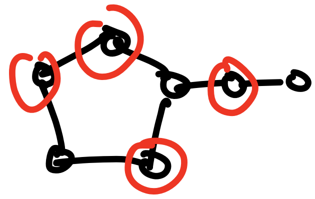
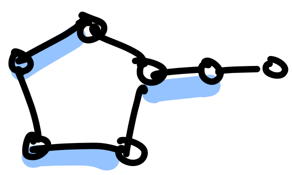


$\nu(G) := \max\{ |M| : M \subset E$ a (partial) matching $\}$

the focus of matching theory

i.e. no $x \in V$ lies in more than one $e \in M$

$= 3$



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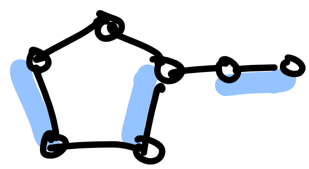


There are some surprising relations ...

THEOREM: (Gallai 1958) If G has no isolated vertices, then

(a) $\underbrace{\nu(G)}_{\text{max matching}} \le \underbrace{\tau(G)}_{\text{min vertex cover}}$, e.g. $\overset{\nu}{3} \le \overset{\tau}{4}$

(b) $\underbrace{\alpha(G)}_{\text{max independent set}} \le \underbrace{\rho(G)}_{\text{min edge cover}}$ e.g. $\overset{\alpha}{3} \le \overset{\rho}{4}$

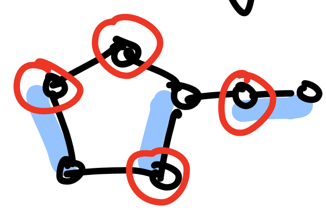
and one has

$$\alpha(G) + \tau(G) \overset{(c)}{=} |V| \overset{(d)}{=} \nu(G) + \rho(G).$$

e.g. $3+4=7=3+4$

Hence one has equality in (a) $\iff$ equality in (b)

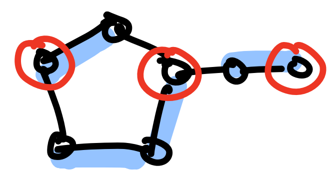
Proof: For (a), note every vertex cover $W \subset V$ needs at least one endpoint from every edge e in any matching $M \subset E$:



$$|M| \le |W| \quad \forall M, W$$

so $\max \le \min$.

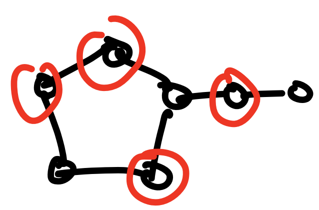
For (b), note every edge cover $F \subset E$ needs at least one edge incident to every vertex x in any independent set $C \subset V$:



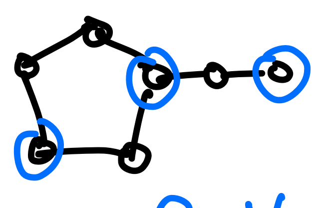
$$|C| \le |F| \quad \forall C, F$$

so $\max \le \min$.

For (c), note that a subset $W \subset V$ is a vertex cover $\iff$ $C := V \setminus W$ is an independent set:



W



$C = V \setminus W$

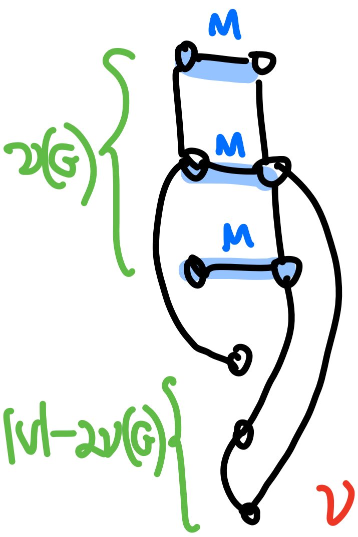
$$|W| = |V| - |C|$$

so $\underbrace{\max\{|W|\}}_{\tau(G)} = |V| - \underbrace{\min\{|C|\}}_{\alpha(G)}$

For (d), we'll show the two inequalities:

$\nu(G)+\rho(G)\leq |V|$: Pick a max size matching $M\subset E$, so $\nu(G)=|M|$. For each of the $|V|-2\nu(G)$ vertices x unmatched by M, add an edge incident to x, giving an edge cover F with

$$|F| \leq \nu(G) + (|V|-2\nu(G)) = |V|-\nu(G).$$



So $\rho(G) = \max\{|F| : \text{edge covers } F\} \leq |V|-\nu(G)$,

i.e. $\nu(G)+\rho(G)\leq |F|$

$\nu(G)+\rho(G)\geq |V|$: Pick a min size edge cover $F\subset E$, so $\rho(G)=|F|$. For each $x\in V$, delete $\deg_F(x)-1$ edges of F incident to x, obtaining a matching M with

$$\begin{aligned} |M| &\geq \rho(G) - \sum_{x\in V}(\deg_F(x)-1) \\ &= \rho(G) - \sum_{x\in V}\deg_F(x) + |V| \\ &= \rho(G) - 2\rho(G) + |V| = |V|-\rho(G) \end{aligned}$$

Hence $\nu(G) = \max\{|M| : \text{matchings } M\}$

$\geq |V| - \rho(G)$

i.e. $\nu(G)+\rho(G)\geq |V|$ ■

CONCLUSION:

We only need to compute one out of $\alpha(G), \tau(G)$, [max indep set] [min edge cover]

one out of $\nu(G), \rho(G)$. [max matching] [min vertex cover]

If turns out that computing $\alpha(G)$ is NP-complete for general graphs G, on Karp's list (1972) of 21 NP complete problems.

On the other hand, finding a max size matching M and hence computing $\nu(G)$ can be done in polynomial time.

We'll also see that the theory is easier for bipartite G, where we'll show

$$\nu(G) = \tau(G)$$

$$\alpha(G) = \rho(G) = |V| - \tau(G)$$

$$= |V| - \nu(G)$$

so that all 4 can be computed in polynomial time.

Q: How to tell when a matching $M \subset E$ is not maximum-sized, i.e. $|M| < \nu(G)$?

One obvious way it can happen:

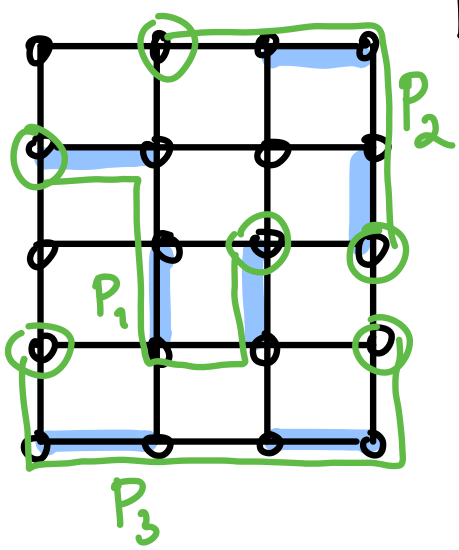
DEFINITION: Given $G = (V, E)$ and a matching $M \subseteq E$, a path P in G is

- **M-alternating** if it alternates edges in/not in M,
- **M-augmenting** if additionally its endpoints are M-unmatched.

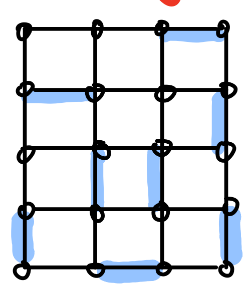
Given an M-augmenting path P, one augments M along P by replacing M with $M' := M \,\Delta\, P$, i.e. swapping edges of P that are in/not in M.

$\boxed{\text{NOTE: } |M'| = |M| + 1}$

EXAMPLE:



In G, M here, the paths P_1, P_2, P_3 are all M-alternating, but only P_3 is M-augmenting.

$M' = M \,\Delta\, P =$ 

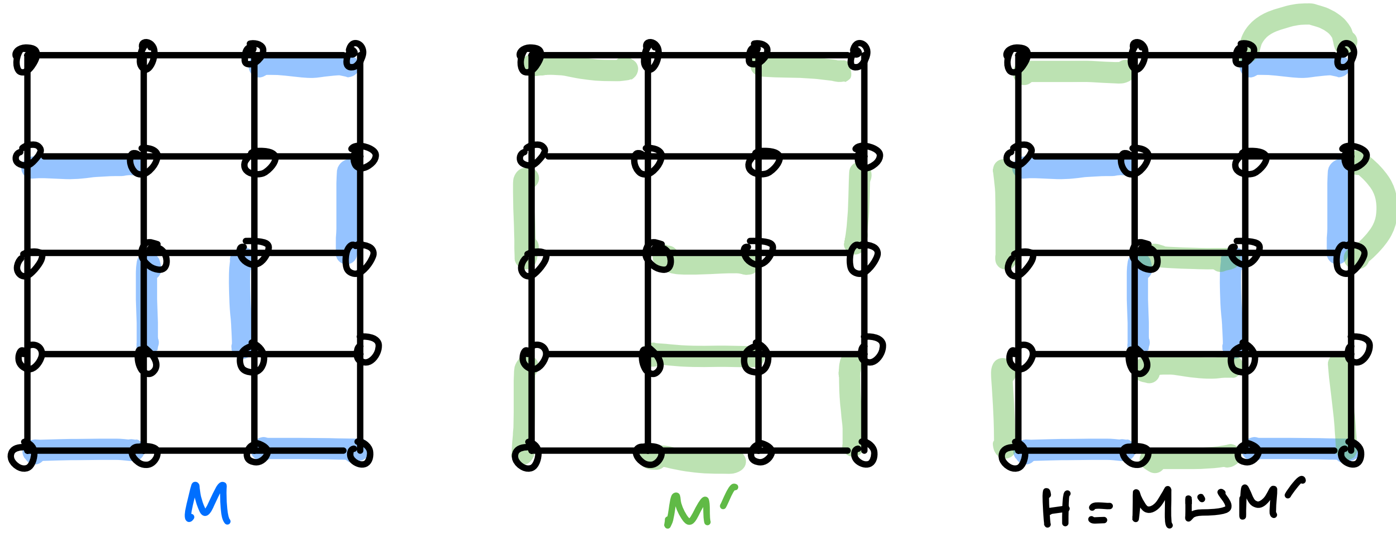
A very surprising (and useful)...

PROPOSITION: For any graph $G=(V,E)$
(Berge 1957)

a matching $M \subset E$ is max-sized
$\iff$ $\nexists$ any M-augmenting paths P.

proof: ($\Rightarrow$): This should be clear, since an M-augmenting path P would give $M' = M \Delta P$ with $|M'| = |M| + 1$.

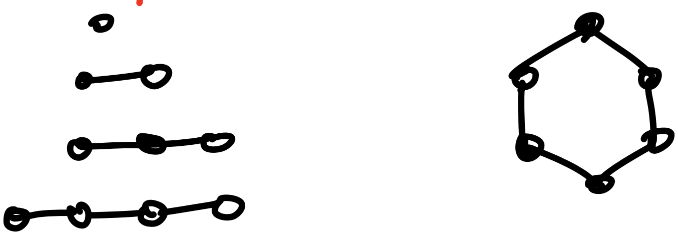
($\Leftarrow$): Suppose M is not max-sized, so $\exists$ some matching M' with $|M'| > |M|$.
We'll show why there must exist some M-augmenting path P. Consider the multigraph $H = (V, M \sqcup M')$, in which every $x \in V$ has $\deg_H(x) \le 2$.



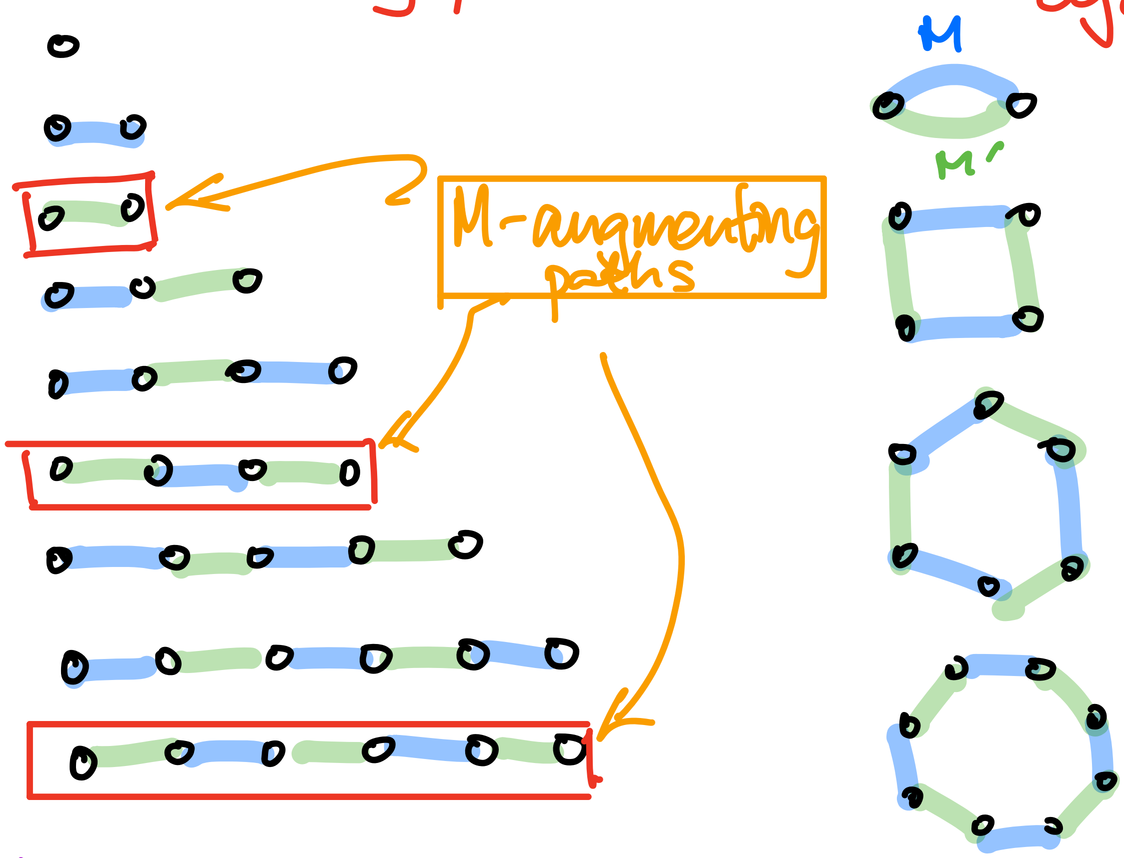
ACTIVE LEARNING: Explain why ...

(a) Every $x \in V$ has $\deg_H(x) \in \{0,1,2\}$

(b) Every connected component is either a path or a cycle



(c) In fact, they must all be M-alternating paths and M-alternating even cycles:

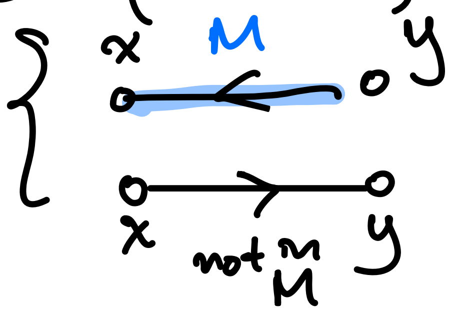


(d) $|M'| > |M| \Rightarrow$ at least one is an M-augmenting path. ∎

Q: OK, so how do we algorithmically **find** M-alternating paths, or **show none exist**?

It's easier (and faster) in the bipartite case...

PROPOSITION: For $G=(V, E)$ bipartite, $V = X \sqcup Y$, and $M \subset E$ any matching, create a digraph $D=(X \sqcup Y, A)$ where arcs go



Then $\left\{ \begin{array}{c} M\text{-augmenting} \\ \text{paths } P \text{ in } G \end{array} \right\} = \left\{ \begin{array}{c} \text{directed paths} \\ P \text{ in } D \text{ from} \\ M\text{-unmatched} \\ \text{vertices } x \text{ in } X \\ \text{to } M\text{-unmatched} \\ \text{vertices } y \text{ in } Y \end{array} \right\}$.

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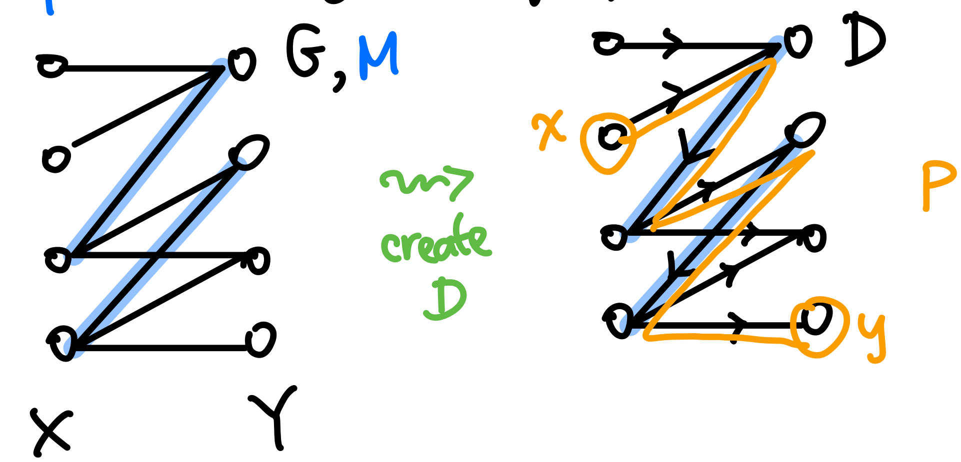
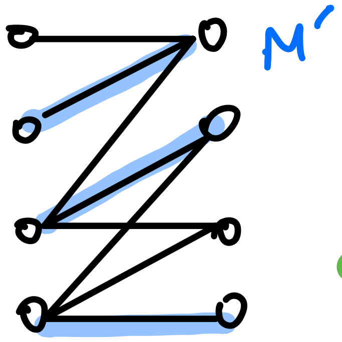
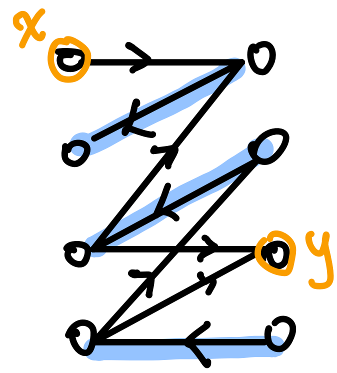
proof: (by example)



augment along P



create, new D′



no paths from any unmatched $x \in X$ to any unmatched $y \in Y$

$\Downarrow$

M' is max-sized; $\nu(G) = 3$. ∎

This gives us Kuhn's "Hungarian algorithm" (after König and Egerváry) to find a max-sized matching in bipartite graphs G:

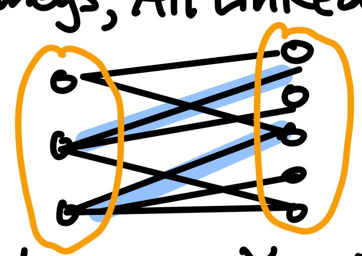
1955

- Start with $M_0 = \emptyset$
- Using M_i, direct G to get D_i
- If $\exists$ directed paths P in D_i from M_i-unmatched $x \in X$ to M_i-unmatched $y \in Y$,

  use P to augment M_i creating M_{i+1}.

  Repeat.
- Otherwise, stop, since M_i is max-sized.

This algorithm helps optimize kidney transplants - see NYTimes Feb 18, 2012 "60 Lives, 30 Kidneys, All Linked" about a long M-augmenting path in

$G =$ 

X = kidney donors $\quad$ Y = recipients

It also has a theoretical consequence:

COROLLARY (König–Egerváry 1931) For bipartite G, $\underset{\text{max matching}}{\nu(G)} = \underset{\text{min vertex cover}}{\tau(G)}$,

and hence also $\underset{\text{max indep. set}}{\alpha(G)} = \underset{\text{min edge cover}}{\rho(G)}$ $\left(= |V| - \nu(G) = |V| - \tau(G) \right)$.

proof: We've already seen $\nu(G) \le \tau(G)$.

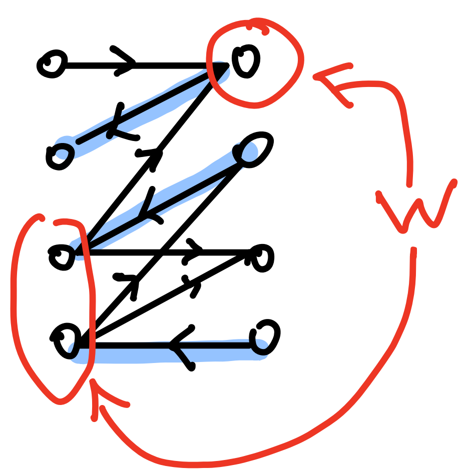
So given a max-sized matching M in G, it's enough to exhibit a vertex cover $W \subset V$ with $|W| \le |M|$, since that shows $\tau(G) \le \nu(G)$.

Use M in $G = (X \sqcup Y, E)$ to create digraph D as before,

and let $W := \left\{ x \in X : \begin{array}{c} \text{not reachable} \\ \text{in } D \text{ from} \\ \text{the } M\text{-unmatched} \\ X \text{ vertices} \end{array} \right\} \sqcup \left\{ y \in Y : \begin{array}{c} \text{reachable} \\ \text{from the} \\ M\text{-unmatched} \\ X \text{ vertices} \end{array} \right\}$

EXAMPLE:

G, M



We claim W is a vertex cover, since

- for edges $x \longrightarrow y$ not in M, if $x \notin W$ then x is reachable, so y is reachable, and then $y \in W$.
- for edges $x \longleftarrow y$ in M, if $x \notin W$ then x is reachable, so it must be reached through y, so y is reachable, and then $y \in W$.

To see that $|W| \leq |M|$, first note that every vertex in W lies on some edge of M:

- if $y \in W$ and $y \in Y$, then y must be matched in M by maximality of M (since y is reachable)
- if $x \in W$ and $x \in X$, then x not reachable implies in particular that x is not M-unmatched, i.e. x is M-matched!

However, note that two vertices of W cannot lie on the same edge of M: $x \longleftarrow y$

unreachable! reachable!

Hence we get an injective map
$$W \longrightarrow M$$
$$v \longmapsto \text{unique edge of } M \text{ on which it lies}$$

and $|W| \leq |M|$ ∎

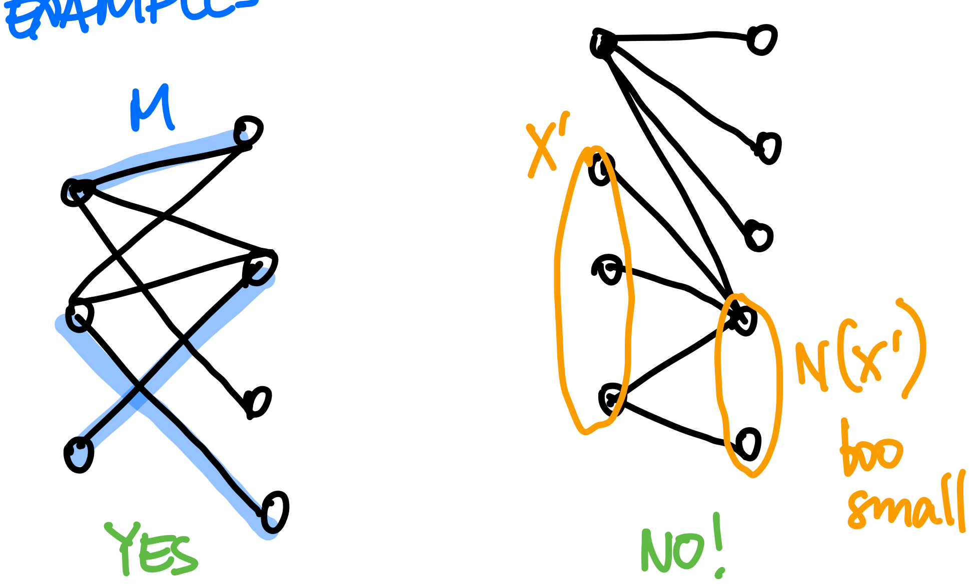
This has an interesting consequence.

COROLLARY (P. Hall's "Marriage Theorem" 1935)

A bipartite graph $G=(V,E)$, $V = X \sqcup Y$, has a matching $M \subseteq E$ that matches all of X

$\iff \forall X' \subseteq X, \quad |\underbrace{N(X')}_{\substack{:= \text{neighbors of } X' \text{ in } G \\ = \{y \in Y :\, \exists e=\{x,y\} \in E\}}}| \geq |X'|$

EXAMPLES



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proof: $(\Rightarrow)$: If M matches all of X, then $\forall X' \subset N(X')$ the map $X' \hookrightarrow N(X')$ is injective

$x \longmapsto$ unique $y \in Y$ with $\{x,y\} \in M$

showing $|X'| \leq |N(X')|$

($\Leftarrow$): If no matching M matches all of X,

then $|X| > \underset{\text{max matching}}{\rho(G)} = \underset{\text{min vertex cover}}{\tau(G)}$ by König-Egerváry Thm.

so $\exists$ a vertex cover $W \subset V$ with $|W| < |X|$.

Let $X' := X - W$.

Then every $y \in N(X')$ lies in W: $x \in X' \Rightarrow y \in W$

Hence $W \supseteq \underbrace{X \cap W}_{\subset X} \sqcup \underbrace{N(X')}_{\subset Y}$

$$\Rightarrow |X| > |W| \geq |X \cap W| + |N(X')|$$
$$= |X| - |X'| + |N(X')|$$

i.e. $|X| > |X| - \underbrace{(|N(X')| - |X'|)}_{\text{must be } < 0}$

$\blacksquare$

Hall's Theorem itself has a number of consequences ...

COROLLARY: Let $G = (V, E)$ be a bipartite multigraph and d-regular for $d \geq 1$, where $V = X \sqcup Y$.

Then (a) $|X| = |Y|$

(b) G has a perfect matching M
matches all vertices

(c) In fact, one can write

$$E = M_1 \sqcup M_2 \sqcup \ldots \sqcup M_d$$

as a disjoint union of d perfect matchings!

ACTIVE LEARNING

We build a 3-regular bipartite multigraph $G = (V, E)$ for $|X| = |Y| = 4$, with audience participation, where $V = X \sqcup Y$.

Then you decompose $E = M_1 \sqcup M_2 \sqcup M_3$ for 3 perfect matchings M_i.

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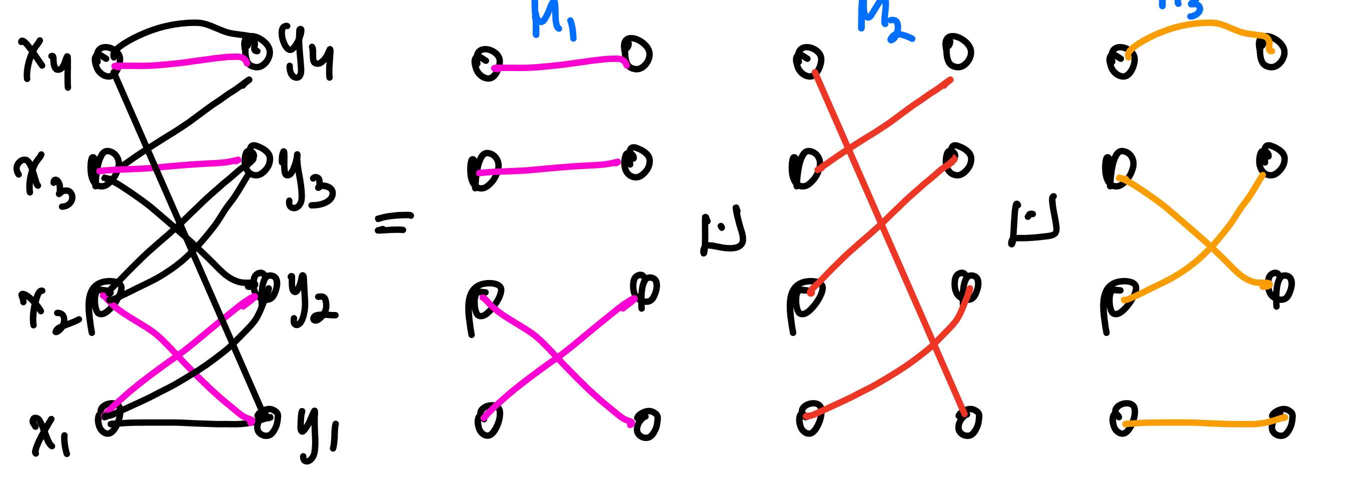
e.g.

$G =$



proof: (a) follows from

$$d\cdot|X| = \sum_{x\in X}\deg_G(x) = |E| = \sum_{y\in Y}\deg_G(y) = d\cdot|Y|$$

$$\Rightarrow |X| = |Y|$$

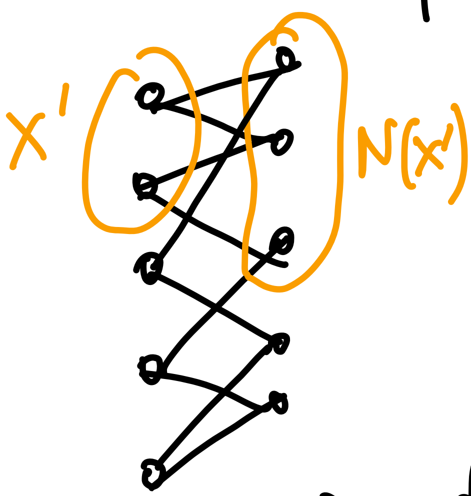
(b) will follow if we can check Hall's hypothesis that $\forall\, X'\subseteq X$, $|N(X')|\geq|X'|$.

Do this by counting two ways

some edges out of $y\in N(X')$ do not go into X'

$$\left|\left\{\begin{matrix}\text{edges } \{x,y\}\in E \text{ from}\\ x\in X' \text{ to } y\in N(X')\end{matrix}\right\}\right| \leq \sum_{y\in N(X')}\deg_G(y) = d\cdot|N(X')|$$

$$\left|\left\{\begin{matrix}\text{edges } \{x,y\}\in E \text{ from}\\ x\in X' \text{ to } y\in N(X')\end{matrix}\right\}\right| = \sum_{x\in X'}\deg_G(x) = d\cdot|X'|$$



$$\Rightarrow d\cdot|X'| \leq d\cdot|N(X')|$$

$$|X'| \leq |N(X')|$$

(c) follows from (b) by induction on d, with easy base case $d=1$ where $G=M$. ∎

REMARK: A similar method applies to prove Bondy & Murty's Exer 5.2.8, the Birkhoff-von Neumann Thm.

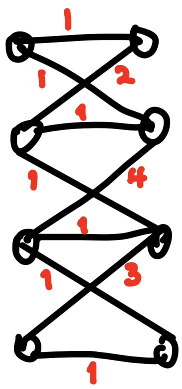
Max weight bipartite matching (Schrijver §3.5, Bondy & Murty §5.5)

Given $G = (X \sqcup Y, E)$ a bipartite graph and weight $w : E \to \mathbb{R}_{\geq 0}$, want to find $M \subset E$ a matching that maximizes $w(M) := \sum_{e \in M} w(e)$.

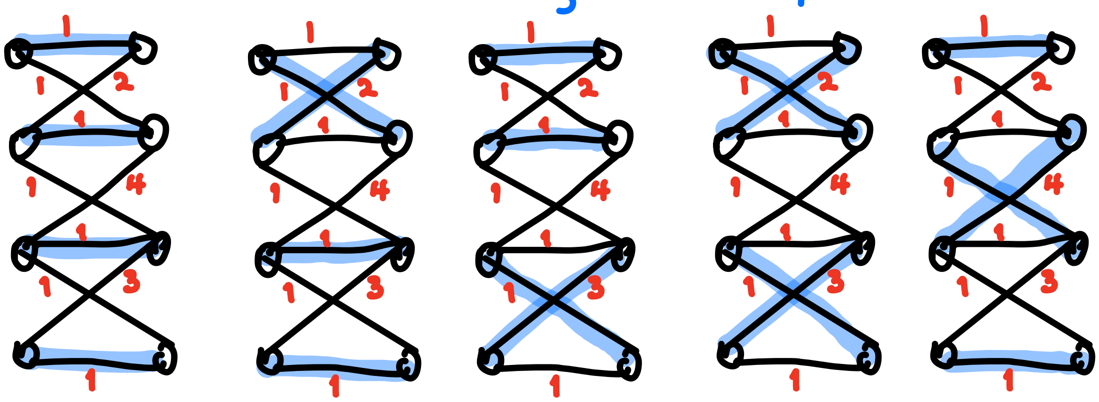
NOTE: It will not always be of size $\nu(M)$!

EXAMPLE:

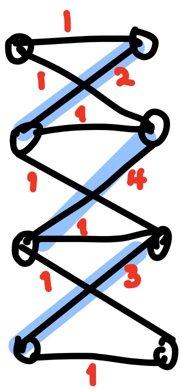
has $\nu(G) = 4$:

$G =$ 

M_1 M_2 M_3 M_4 M_5



M_1	M_2	M_3	M_4	M_5
$1+1+1+1 = 4$	$1+2+1+1 = 5$	$1+1+3+1 = 6$	$1+2+3+1 = 7$	$1+1+4+1 = 7$

but $M =$  has $w(M) = 2+4+3 = 9$ beats them all.

DEFINITION: Call a matching $M \subseteq E$ **extreme** if it has max weight $w(M)$ among all matchings in G of the same size.

EXAMPLE: M, M_4, M_5 are extreme above

Kuhn (1955) gave a fast generalization of his Hungarian algorithm that finds at least one extreme matching $M_0, M_1, M_2, \ldots, M_{\nu(G)}$ of each size $|M_i| = i$ for $i = 0, 1, 2, \ldots, \nu(G)$.

(Then picking whichever M_i maximizes $w(M_i)$ solves the problem)

Kuhn's algorithm:

Start with $M_0 = \emptyset$ as before.

Given M_i, direct G as a digraph $D = (X \sqcup Y, A)$ as before, i.e.

$$\begin{cases} x \longrightarrow y & \text{if } \{x,y\} \notin M \\ x \longleftarrow y & \text{if } \{x,y\} \in M \end{cases}$$

But now use the weights $w : E \to \mathbb{R}_{\leq 0}$ also assign the arcs "lengths" $\ell(a)$:

$$x \xrightarrow{-w(e)} y$$

$$x \xleftarrow{w(e)} y$$

Now try to find a directed path P from an M_i-unmatched $x \in X$ to some M_i-unmatched $y \in Y$.

- If such a P exists, augment M_i along P to obtain M_{i+1}.
- If no such P exists, stop, because $|M_i| = \nu(G)$.

Why does it work?

PROPOSITION: If M_i was extreme, then so is M_{i+1}.

proof: Assuming M_i was extreme, let M'_{i+1} be any extreme matching with $|M'_{i+1}| = i+1$. We want to show $w(M'_{i+1}) \leq w(M_{i+1})$, so M_{i+1} is also extreme.

Note the multigraph $M_i \uplus M'_{i+1}$ contains some M_i-augmenting path P' by our old proof of Berge's Thm. We also know that unaugmenting M'_{i+1} along P' gives a matching M'_i of size i, which therefore must have $w(M'_i) \leq w(M_i)$.

We know $\ell(P') \geq \ell(P)$ by construction.

Note $w(M_{i+1}) = w(M_i) - \ell(P)$

$w(M'_{i+1}) = w(M'_i) - \ell(P')$

Hence $w(M'_{i+1}) = w(M'_i) - \ell(P')$

$\leq w(M'_i) - \ell(P)$

$\leq w(M_i) - \ell(P) = w(M_i)$. ▨

An issue remains: Can one quickly find directed paths in the digraph D of minimum length when some arcs have negative length?

YES; there is an easy breadth-first type of algorithm (called the Bellman-Ford algorithm) 1956 ← see Schrijver §1.3 to find shortest directed paths $x_0 \to \to \dots \to x$ $\forall x \in V$ in a digraph $D = (V, A)$ with arclengths $\ell: A \to \mathbb{R}$, as long as ℓ leads to no directed cycles C in D of negative length $\ell(C) < 0$.

To find shortest directed paths from $x_0 \in V$ to all other $x \in V$ for some $D=(V,A)$ with $\ell : A \to \mathbb{R}$, proceed in stages labeling each x with the shortest path length $\lambda(x)$ reaching it so far, starting with all labels $\lambda(x) = \infty$.

STAGE 0: Label x_0 as $\lambda(x_0) = 0$.

STAGE $i+1$: Proceed along arcs of form $x \xrightarrow{a} y$ where x had $\lambda(x)$ updated in stage i.

Update $\lambda(y) = \min\{\lambda(y), \lambda(x_i) + \ell(a_i) : x_i \xrightarrow{a_i} y\}$ with x_i updated in stage i

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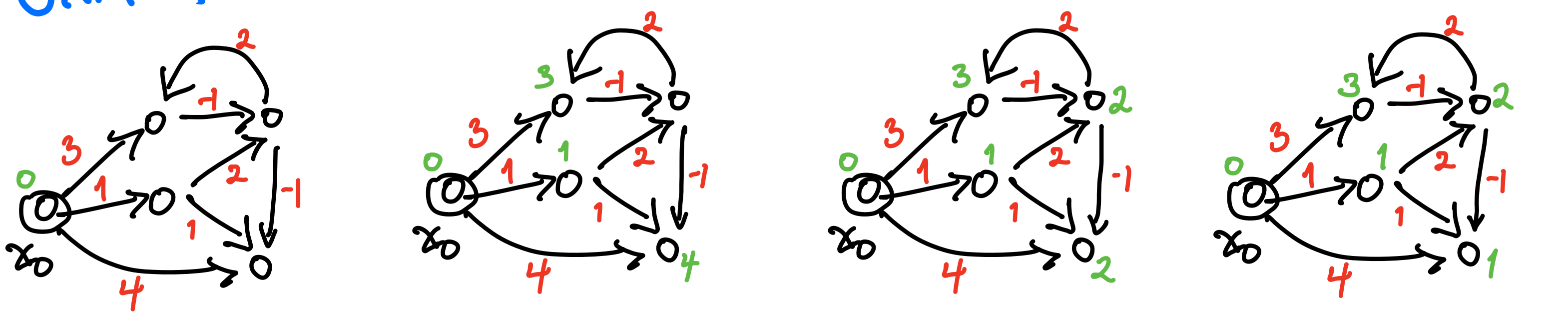
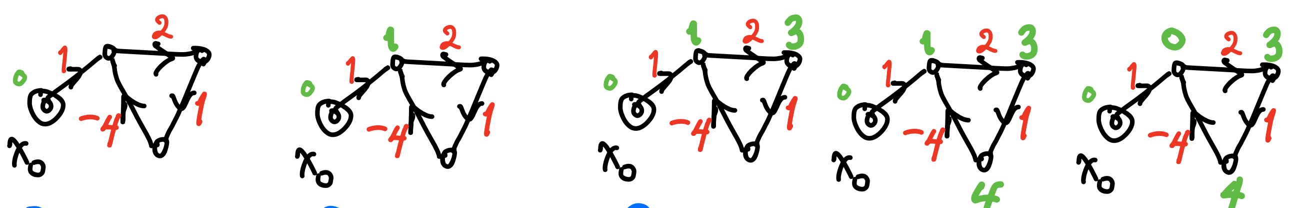
EXAMPLES:



STAGE 0 STAGE 1 STAGE 2 STAGE 3



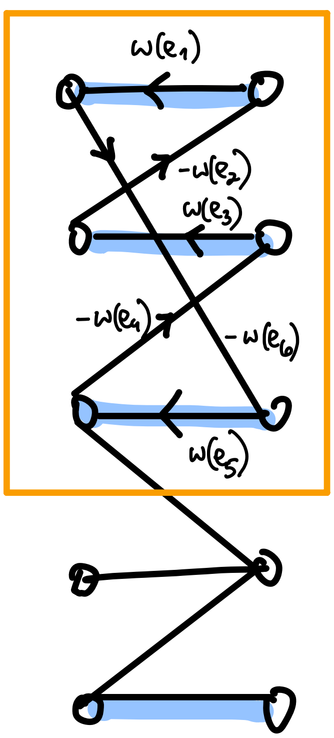
STAGE 0 STAGE 1 STAGE 2 STAGE 3 STAGE 4

. . .
never terminates

LEMMA: During Kuhn's algorithm, the digraphs D_i never have directed cycles C with $\ell(C)<0$.

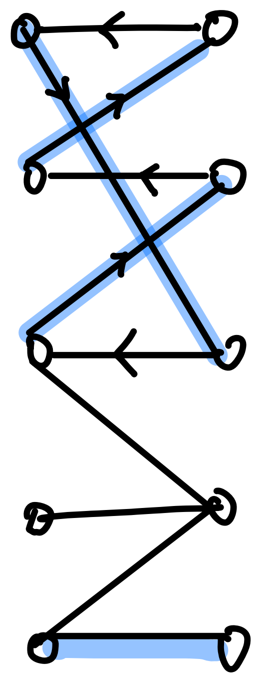
Proof: Given the extreme matching M_i in G, if the digraph D_i it created had such a cycle C, then one could swap M_i's edges in $M_i \cap C$ with those in $C \setminus M_i \cap C$ to get a matching M_i' with $w(M_i') > w(M_i)$. Contradiction ∎

C



M_i

$$\ell(C) = w(e_1)+w(e_3)+w(e_5)-w(e_2)-w(e_4)-w(e_6) < 0$$



M_i'

$$w(M_i') = w(M_i) - \ell(C) > w(M_i)$$

A glimpse of non-bipartite matching theory

We saw these two matching theorems for bipartite G:

THEOREM (König-Egerváry)
G bipartite has $\nu(G) = \tau(G)$
max size matching — min size vertex cover

THEOREM (Hall)
$G = (X \sqcup Y, E)$ bipartite has a perfect matching (all vertices matched)
$\iff$ $|X| = |Y|$ and $\forall X' \subset X$ one has $|N(X')| \geq |X'|$

Both have interesting generalizations to nonbipartite G.

THEOREM (Tutte-Berge formula, 1958) - see Schrijver §5.1

Any simple graph $G = (V, E)$ has

$$\nu(G) = \min_{U \subseteq V} \frac{1}{2}\left(|V| + |U| - \begin{matrix}\#\text{odd connected}\\ \text{components of } G \setminus U\end{matrix}\right)$$

THEOREM (Tutte's 1-factor Theorem, 1947) (Bondy & Murty Thm. 5.4)

Any simple graph $G = (V, E)$ has a perfect matching
$\iff$ $\forall U \subseteq V$, $\#$odd connected components of $G - U \leq |U|$

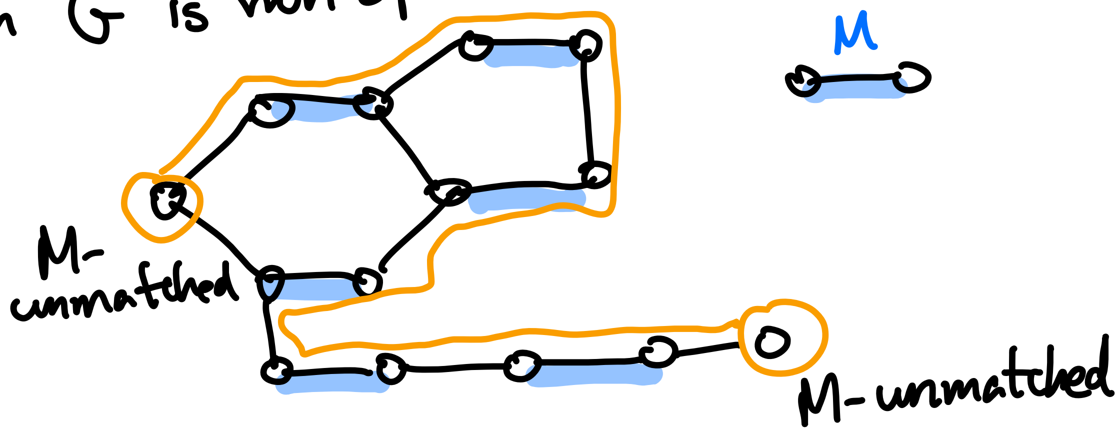
One can show Tutte-Berge Formula $\Rightarrow$ Tutte's 1-factor Thm.

Showing that they imply their bipartite special cases also takes a little thought! (Skipped here.)

Edmond's blossom algorithm for max-sized matching in nonbipartite G

Q: How to find M-augmenting paths P in G when G is non bipartite?

e.g.

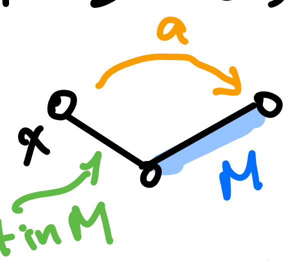


A: Finding shortest directed paths in a certain digraph D are still relevant:

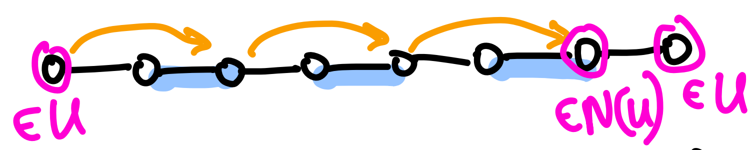
PROPOSITION: (not hard) Let $\mathcal{U} :=$ M-unmatched vertices of G

$N(\mathcal{U}) :=$ neighbors of $\mathcal{U}$

and create a digraph $D = (V, A)$ where A has arcs a like this:



Then shortest paths from $\mathcal{U}$ to $N(\mathcal{U})$ either

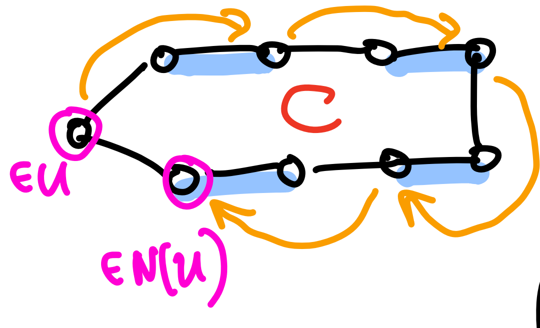
- are M-augmenting paths 

(if they never revisit vertices or hop over visited vertices)

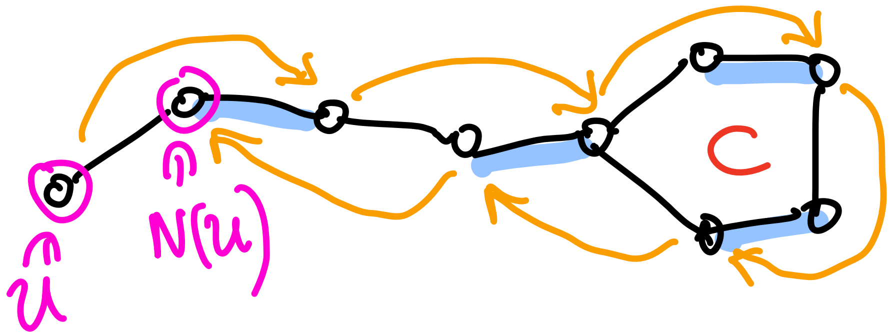
(GOOD news – lets us augment M along path)

- or give rise to what Edmond called

blossoms := odd cycles C with this matching pattern:

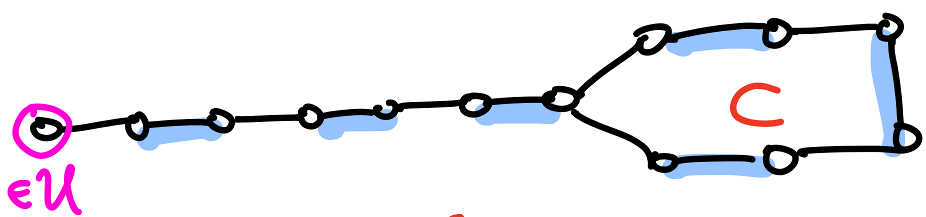


(Seems like BAD news ?!)

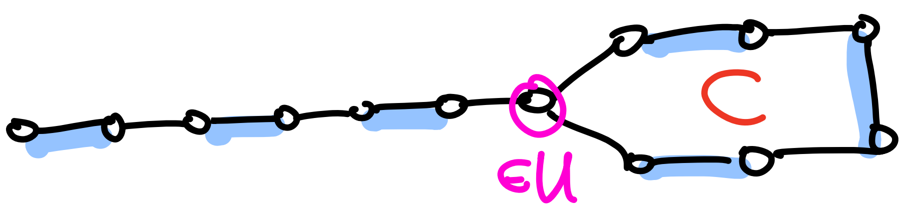


Q: How to eliminate the blossoms we encounter?

FIRST, make sure the blossom's cycle C has an M-unmatched vertex x_0, by shifting the matching M along the stem:



shift M along the stem



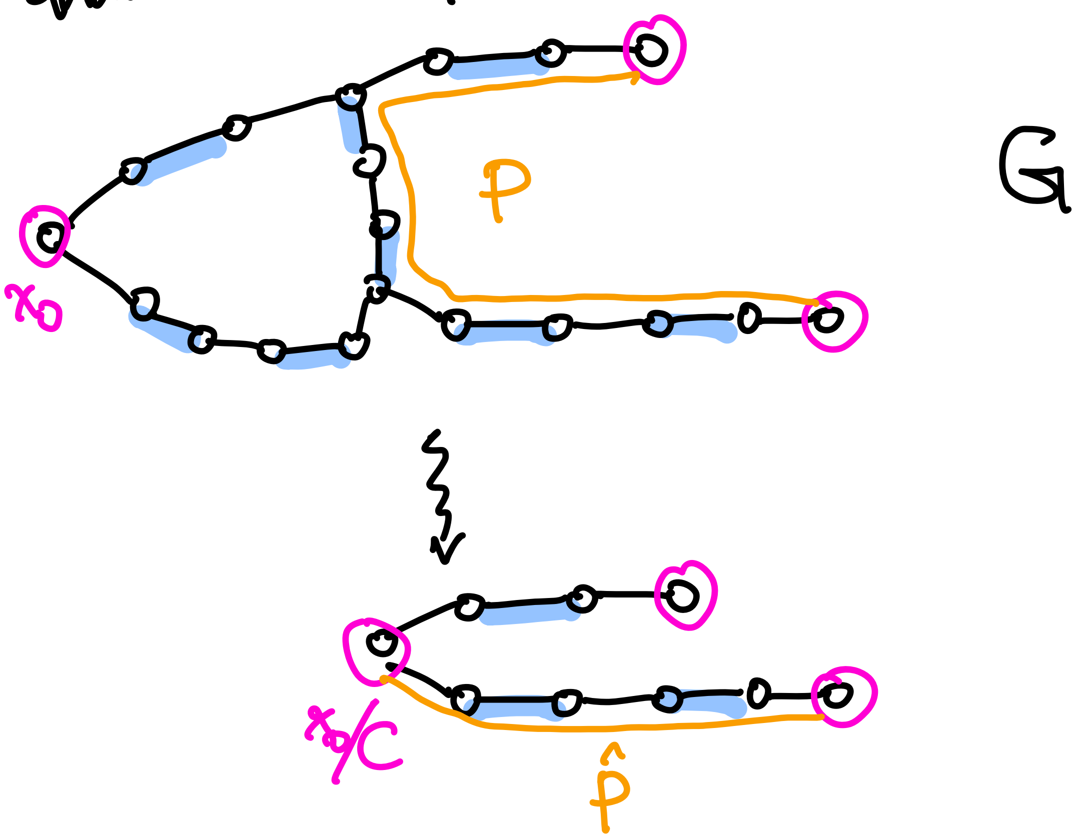
SECOND, contract down C to a single vertex x_0/C, forming G/C with matching M/C.

Then apply this fact:

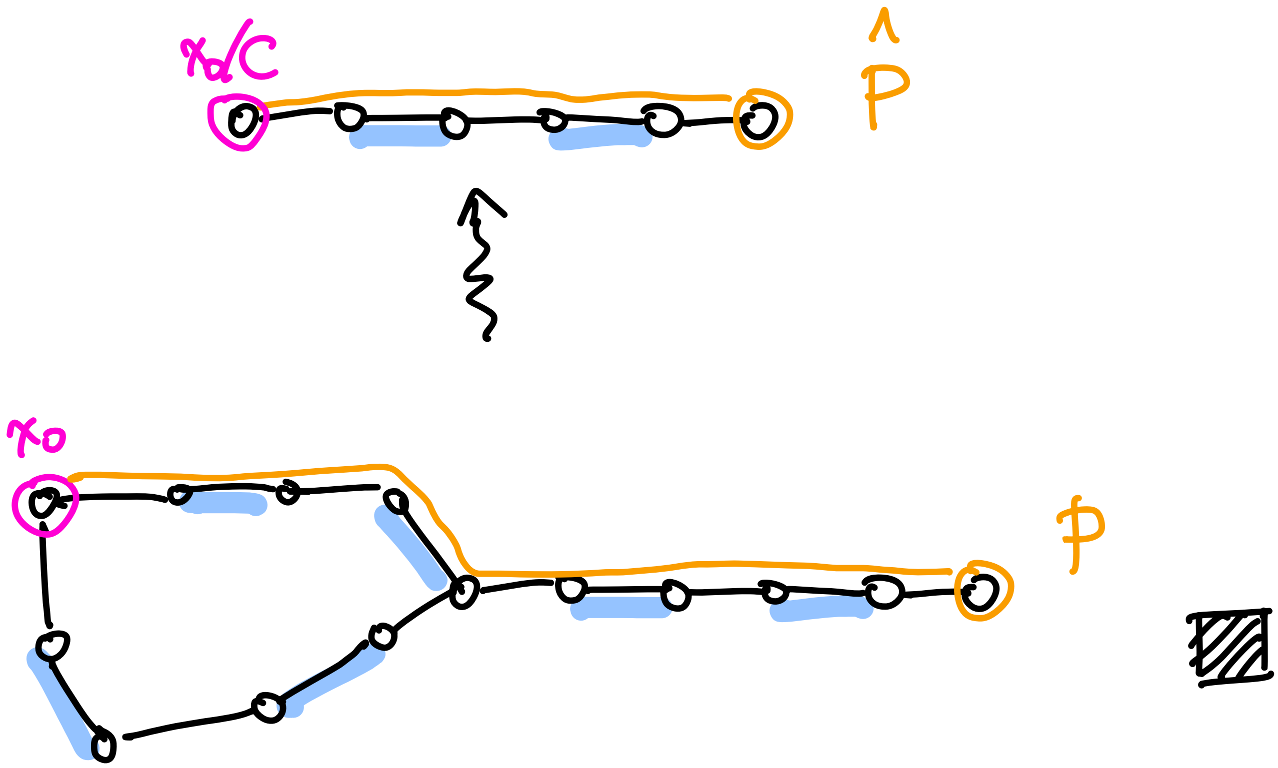
PROPOSITION: G has an M-augmenting path P $\iff$ G/C has an M/C-augmenting path $\hat{P}$.

proof idea:

($\Rightarrow$): An M-augmenting path P in G either misses C entirely, so it persists in G/C, or P hits C and enters it along a non-M edge, so that G/C has an M-augmenting path $\hat{P}$ that ends at x_0/C:



($\Leftarrow$): An M/C-augmenting path $\hat{P}$ in G/C either misses x_0/C entirely, so it persists in G, or $\hat{P}$ ends at x_0/C and there is exactly one way to expand it to an M-augmenting P in G that ends at x_0:



This gives Edmond's Blossom Algorithm (1961) to compute $\nu(G)$ and find a max-sized matching M in $G=(V,E)$ in $\leq c \cdot |V| \cdot |E|^2$ steps.
At each step, it runs a depth-first search for a $U - N(U)$ M-alternating path P, which is either M-augmenting, or it finds a blossom C to contract, and then works in G/C.

(see Schrijver Chap. 5)