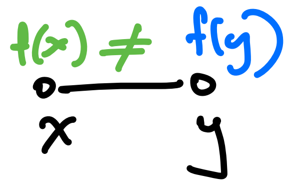
Vertex, edge colorings and perfect graphs (Schrijver §7.4)

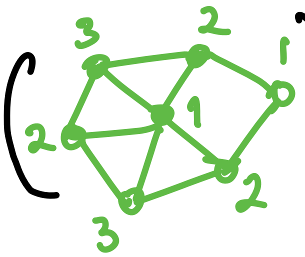
(Bondy-Murty Chap. 8, Chap. 6)

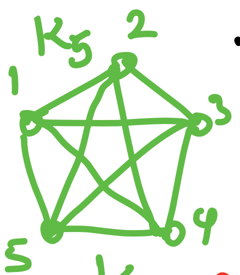
DEFINITION: Given $G=(V,E)$ a (simple) graph, an assignment $f: V \to \{1,2,\dots,k\}$ is called a *proper (vertex) k-coloring* if $f(x)\neq f(y)$ $\forall$ edges $e=\{x,y\}\in E$.



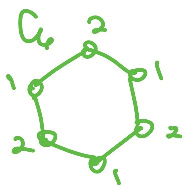
$\chi(G)$:= chromatic number of G
:= $\min\{k : \exists \text{ a proper } k\text{-coloring of } G\}$

EXAMPLES:

① $\chi\Big($  $\Big) = 3$

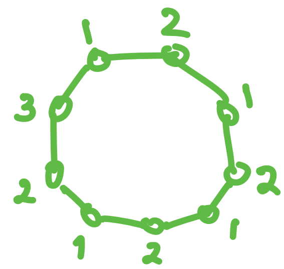
② $\chi\Big($  $\Big) = n$ for $n \geq 1$

K_n complete graph



③ $\chi(C_n) = \begin{cases} 2 & \text{if } n \text{ is even} \\ 3 & \text{if } n \text{ is odd} \end{cases}$

n-cycle



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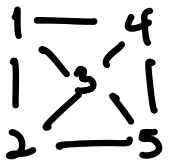
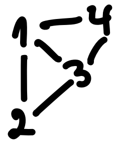
④ $\chi(G) \geq \chi(G[V'])$ for any $V' \subseteq V$

vertex-induced subgraph of G on V'
$:= (V', E')$
where $E' = \{e = \{x,y\} \in E : x, y \in E'\}$

e.g.

$G =$ 

has $G[\{1,2,3,4\}] =$ 

In particular, $\chi(G) \geq \omega(G)$ = max clique size
$:= \max\{k : K_k \cong G[V']$ for some $V' \subseteq V\}$

⑤ For fixed $k = 3, 4, 5, \ldots$
deciding whether $\chi(G) \leq k$ is NP-complete.
In fact, deciding $\chi(G) \leq 3$ was on Karp's 1972 list of 21 NP-complete problems.

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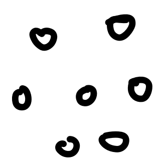
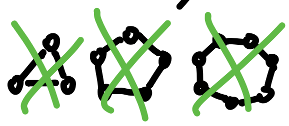
⑥ ACTIVE LEARNING: Prove these:

(a) $\chi(G) = 1 \iff G$ has no edges, i.e. $E = \emptyset$



(b) $\chi(G) \leq 2 \iff G$ is bipartite
$\iff G = (V, E)$ contains no odd cycles

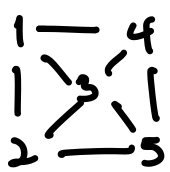
 ... i.e. $C_{2k+1} \not\subseteq E$

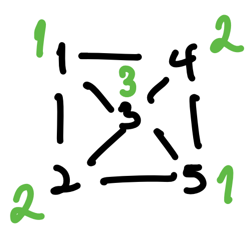
One can get an easy upper bound on $\chi(G)$ in terms of vertex degrees from the greedy coloring algorithm: Order $V = \{x_1, x_2, \ldots, x_n\}$ and then for $i = 1, 2, \ldots, n$ assign vertex x_i color

$$f(x_i) := \min\left(\{1,2,3,\ldots\} \setminus \left\{ f(x_j) : \begin{array}{l} j \in \{1,2,\ldots,i-1\} \\ \{x_i, x_j\} \in E \end{array} \right\} \right)$$

I.e., x_i gets assigned the smallest available color not used by any of its neighbors among $\{x_1, x_2, \ldots, x_{i-1}\}$

EXAMPLE

$G =$  with V ordered $1,2,3,4,5$ gets greedy coloring



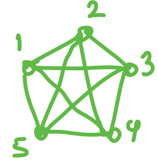
$\Rightarrow \chi(G) \leq 3.$

COROLLARY:

$$\chi(G) \leq 1 + \max\left\{ \deg_{G[\{x_1, x_2, \ldots, x_{i-1}\}]}(x_i) : i = 1, 2, \ldots, n \right\}$$

$$\leq 1 + \underbrace{\Delta(G)}_{:= \text{max vertex degree in } G}$$

We saw $\chi(K_n) = n = 1 + \Delta(K_n)$



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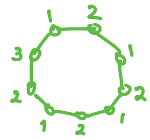
$\chi(C_n) = 3 = 1 + \Delta(C_n)$
n odd



C_9

Surprisingly, in all other cases one can do slightly better:

THEOREM (Brooks 1947) For a connected simple graph G unless $G = K_n$ or $G = C_n$ for n odd,

$$\chi(G) \leq \Delta(G)$$ = max vertex degree

proof: Assume $G \neq K_n$ and $G \neq C_n$ for n odd.

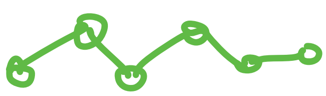
We'll show $\chi(G) \leq \Delta(G)$ by induction on $|V|$.

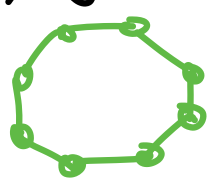
CASE 1: $\Delta(G) = 1$.

Then G connected $\Rightarrow$ $G =$  $= K_2$ so $\chi(G) = 2$. ✓

CASE 2: $\Delta(G) = 2$.

Then G connected $\Rightarrow$ G is a path or (even) cycle





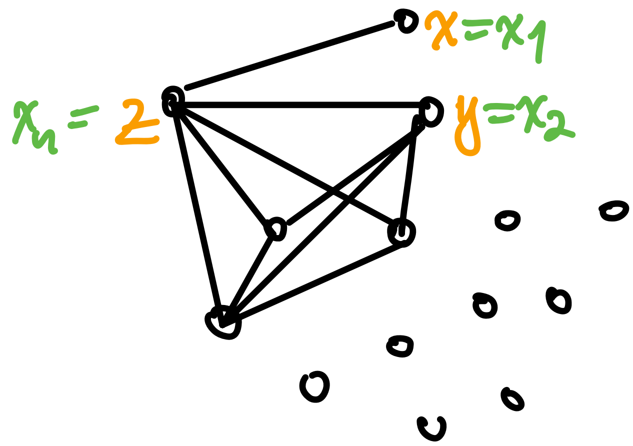
$\chi(G) = 2$ ✓ $\chi(G) = 2$ ✓

CASE 3: $\Delta(G) \geq 3$.

SUBCASE 3a: $\forall$ non-edges $\{x,y\} \notin E$, the graph $G - \{x\} - \{y\}$ is still connected.

Pick $z \in V$ achieving $\deg_G(z) = \Delta(G)$, and then find 2 neighbors x, y of z in G with $\{x,y\} \notin E$

(such x,y exist or else $G = K_{\Delta(G)+1}$)



Color G via greedy coloring, using order like this:

$$\underset{x}{\overset{\shortparallel}{x_1}},\ \underset{y}{\overset{\shortparallel}{x_2}},\ \Big|\ x_3,\ x_4,\ \ldots\ \Big|\ \ldots,\ x_{n-1},\ \underset{z}{\overset{\shortparallel}{x_n}}$$

connected, by assumption

number so that each of these final segments is also connected in G

(pick a spanning tree T in $G[\{x_3, x_4, \ldots, x_n\}]$ and pluck off leaves $x_3 \neq x_n$, $x_4 \neq x_n$, $\vdots$)

Then $f(x)=1=f(y)$,

$f(x_j)\in\{1,2,\dots,\Delta(G)\}$ for $j=3,4,\dots,n-1$

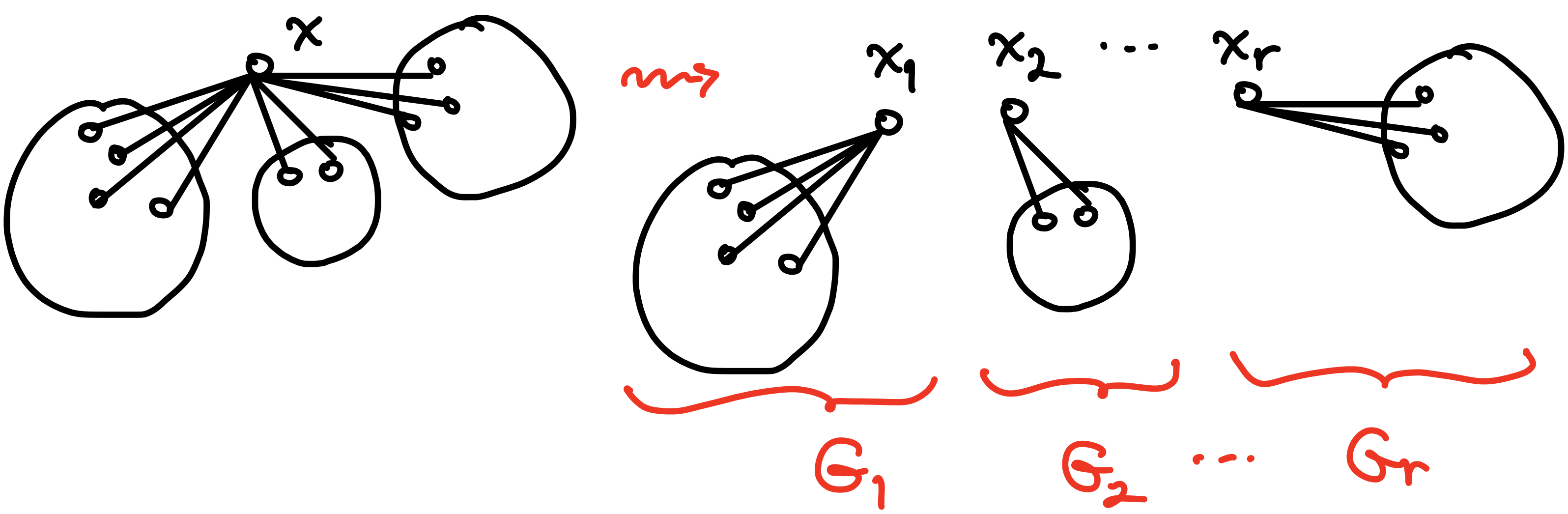
since $\deg_{G\{x_1,x_2,\dots,x_{j-1}\}}(x_j) \le \Delta(G)-1$ because

x_j has some neighbor among $\{x_{j+1},x_{j+2},\dots,x_n\}$

and finally $f(z)\le\Delta(G)$ since its neighbors

x,y have $f(x)=1=f(y)$.

SUBCASE 3b: G has a cut vertex $x\in V$.



Each of $G_1,G_2,\dots,G_r$ has $\chi(G_i)\le\Delta(G)$ by induction (even if some $G_i\cong K_s$,

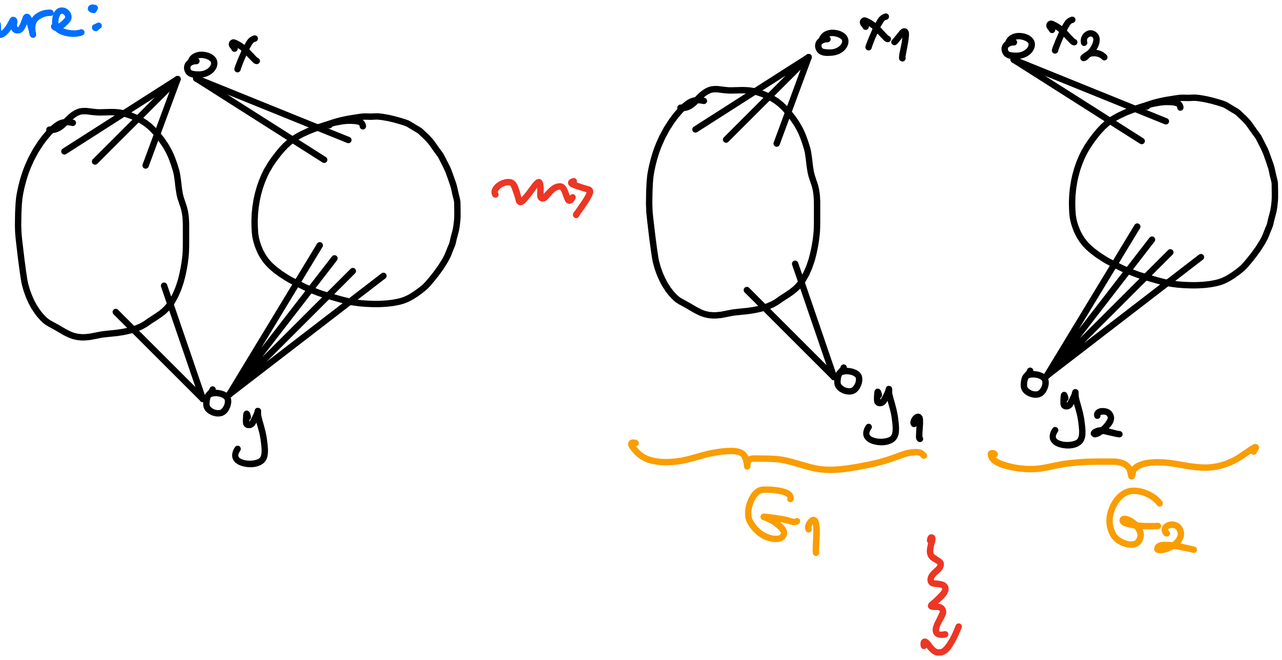
since then $s\le \deg_G(x_i)-1\le\Delta(G)-1$)

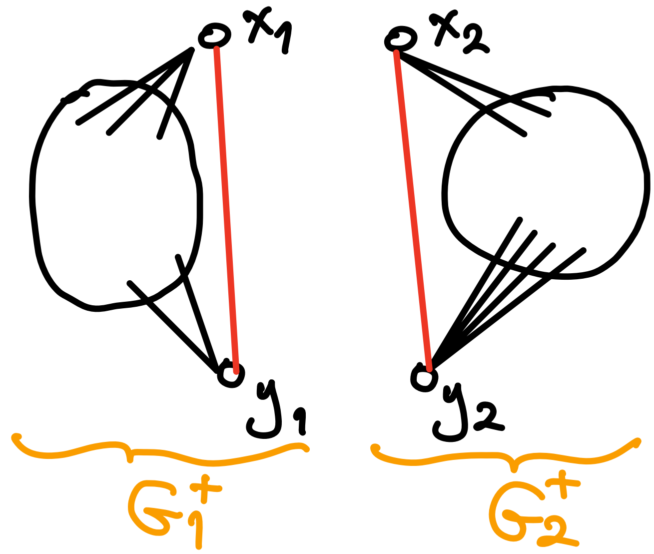
Change the color names in a proper $\Delta(G)$-coloring of each $G_1,G_2,\dots,G_r$ so that they agree on color of x_i, and glue them to get a proper $\Delta(G)$-coloring of G.

SUBCASE 3c: G has no cut-vertex, that is, it is 2-vertex-connected, but has a non-edge $\{x,y\} \notin E$ with $G - \{x\} - \{y\}$ disconnected.

Picture:

$G =$





Note
$\deg_{G_1}(x_1), \deg_{G_2}(x_2),$
$\deg_{G_1}(y_1), \deg_{G_2}(y_2) \geq 1,$
otherwise x or y would be a cut-vertex in G.
$\Rightarrow$ induction on $|V|$ applies to G_1^+, G_2^+, and they have proper $\Delta(G)$-colorings, with $f(x_i) \neq f(y_i)$, so they can be re-colored and glued together, UNLESS...

one of G_1^+, G_2^+ is a complete $K_{\Delta(G)+1}$

(can't have both G_1^+, G_2^+ being cycles, since $\Delta(G) \geq 3$).

If $G_1^+ = K_{\Delta(G)+1}$ then $\deg_{G_2}(x) = 1 = \deg_{G_2}(y)$

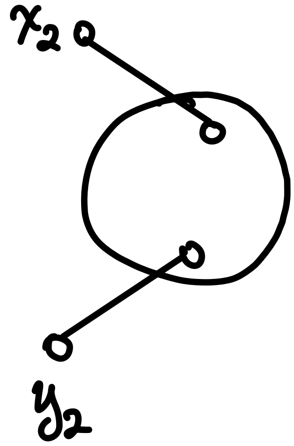
and one can form both of these:

$G_1 =$

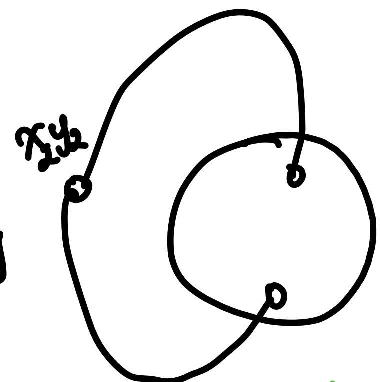
properly $\Delta(G)$-colorable

with $f(x_1) = f(y_1)$

$G_2 =$



$G_2/\{x_2, y_2\}$



properly $\Delta(G)$-colorable, by induction

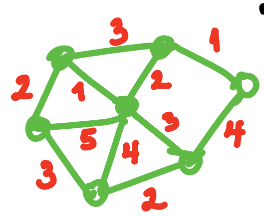
One can re-color and then glue these proper $\Delta(G)$-colorings of G_1 and $G_2/\{x_2, y_2\}$ to get a proper $\Delta(G)$-coloring of G (with $f(x) = f(y)$) ▨

Edge-coloring (Bondy-Murty Chap. 6)

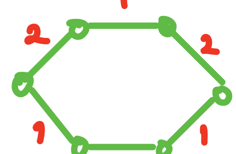
DEFINITION: Given $G=(V,E)$ a loopless multigraph, an assignment $f: E \to \{1,2,3,\dots\}$ is called a proper edge k-coloring if $f(e) \neq f(e')$ $\forall$ edges e, e' incident at some vertex v.

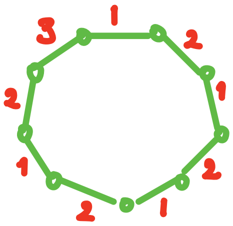
$\chi'(G) :=$ edge chromatic number of G
$= \min\{k : \exists \text{ a proper edge } k\text{-coloring of } G\}$

EXAMPLES:

① $\chi'\Bigg($  $\Bigg) = 5$

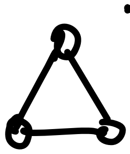
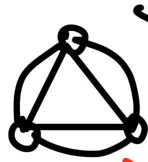
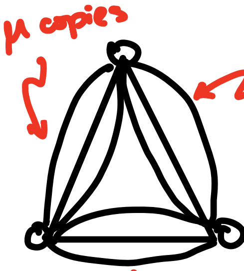
② $\chi'(C_n) = \begin{cases} 2 & \text{if } n \text{ is even} \\ 3 & \text{if } n \text{ is odd} \end{cases}$





③ One can see that $\chi'(G) \geq \underbrace{\Delta(G)}_{\text{max vertex degree}}$

ACTIVE LEARNING Compute

$\chi'\Big($  $\Big)$, $\chi'\Big($  $\Big)$, $\chi'\Big($  $\Big)$.

$(\mu=1)$ $(\mu=2)$

μ copies
2μ copies
μ copies

It is again NP-complete to compute $\chi'(G)$ in general, but even more frustrating due to...

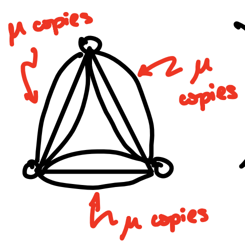
THEOREM (Vizing 1964)
(Bondy-Murty Thm 6.2)

For any simple graph G,

$$\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$$

and more generally, for any multigraph G

$$\Delta(G) \leq \chi'(G) \leq \Delta(G) + \underbrace{\mu(G)}_{\text{max edge multiplicity}}$$

e.g. $\chi'\Big($ 

$\Big) = 3\mu = \underbrace{2\mu}\,+\,\underbrace{\mu}$

$= \Delta(G) + \mu(G)$

Vizing's Theorem is not so hard to prove, but we'll skip it - see Bondy & Murty for the proof.

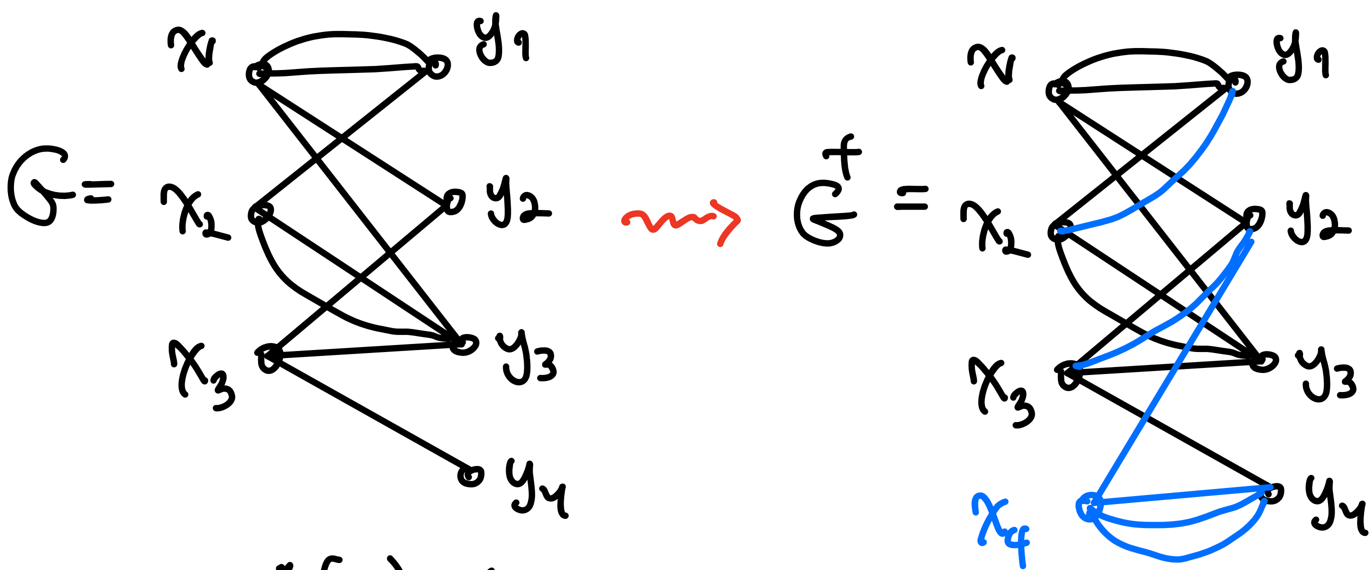
However, let's show that bipartite graphs have more predictable $\chi'(G)$ ...

THEOREM (König's "Line-coloring Theorem" 1937)

For a bipartite multigraph $G = (X \sqcup Y, E)$,

$$\chi'(G) = \Delta(G).$$

proof: Given a bipartite multigraph G, one can add vertices and edges until it is (bipartite and) $\Delta(G)$-regular.



$\Delta(G) = 4$

In this new bipartite $\Delta(G)$-regular graph $G^+ = (V^+, E^+)$, we've seen one can decompose its edge set

$$E^+ = M_1 \sqcup M_2 \sqcup \cdots \sqcup M_{\Delta(G)}$$

into $\Delta(G)$ perfect matchings, which gives a proper edge $\Delta(G)$-coloring of G^+, and restricts to such a $\Delta(G)$-coloring for G. $\blacksquare$

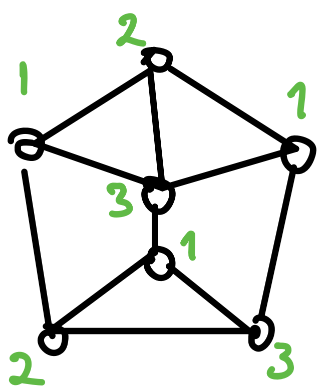
Perfect graphs (Schrijver §7.4)

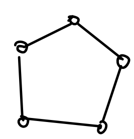
Berge (1963) noted that several notable families of simple graphs $G=(V,E)$

- had equality in the obvious inequality
$$\chi(G) \geq \omega(G) = \text{max clique size}$$
- were closed under taking vertex-induced subgraphs $G[V']$ for $V' \subseteq V$
- and their complement graphs $\overline{G}$ also seemed to have this property

He formulated this as a definition.

DEFINITION: Call a simple graph $G=(V,E)$ perfect if $\forall\, V' \subseteq V$ one has $\chi(G[V']) = \omega(G[V'])$.

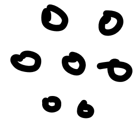
NON-EXAMPLE: $G=$ 

has $\chi(G)=3=\omega(G)$ but contains $G'=G[V']=$ 

with $\chi(G')=3>2=\omega(G)$, so is not perfect.

EXAMPLES:

① Bipartite graphs G are perfect, since either

- $\chi(G) = 1 = \omega(G)$ if G has no edges 

or

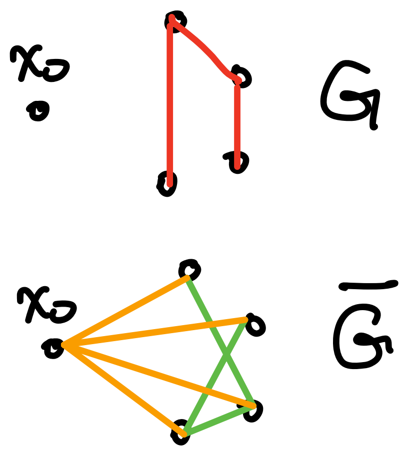
- $\chi(G) = 2 = \omega(G)$

and all of their vertex-induced subgraphs $G[V']$ are also bipartite.

② **PROPOSITION:** Complements $\overline{G}$ of bipartite graphs G are perfect.

<u>proof</u>: Without loss of generality, can assume G has no isolated vertices, since such a vertex x_0 in G leads to a vertex x_0 in $\overline{G}$ connected to all of $V - \{x_0\}$

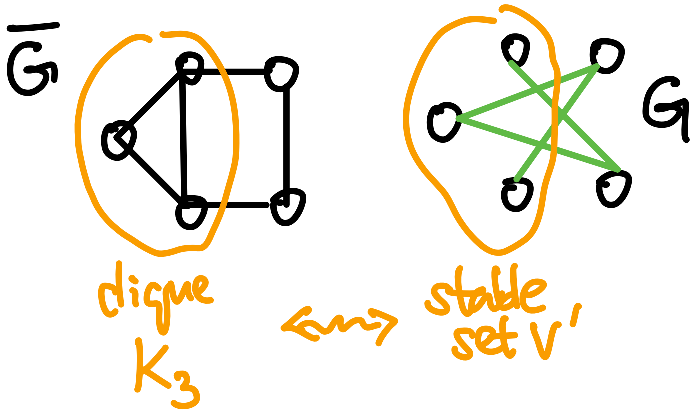
so that
$$\chi(\overline{G}) = 1 + \chi(\overline{G} - \{x_0\})$$
$$\omega(\overline{G}) = 1 + \omega(\overline{G} - \{x_0\})$$



and similarly for all $\overline{G[V']}$.

But then one can note that

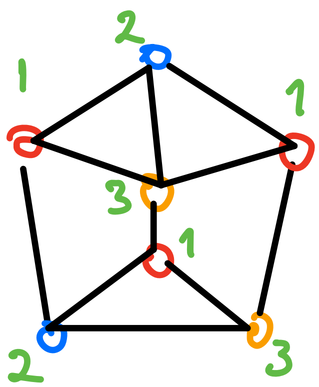
$\omega(\overline{G}) = \alpha(G)$ = max size of an independent/stable set of vertices $V' \subseteq V$ in G



while $\chi(\overline{G}) = \min\{k : \overline{G}$ has a proper vertex k-coloring$\}$

$= \min\{k : V = V_1 \sqcup V_2 \sqcup \dots \sqcup V_k$ with V_i indep. in $\overline{G}\}$

vertices of color 1 — vertices of color 2 — vertices of color k



$= \min\{k : V = V_1 \cup V_2 \cup \dots \cup V_k$ with V_i indep in $\overline{G}\}$

$= \min\{k : V = V_1 \cup V_2 \cup \dots \cup V_k$ with V_i cliques in $G\}$

i.e. either $V_i = \{x\}$ or $V_i = \{x,y\}$ an edge of G since G is bipartite

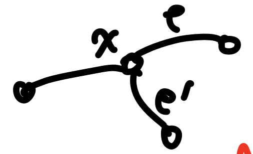
$= \min\{k : V = V_1 \cup V_2 \cup \dots \cup V_k$ with $V_i = \{x,y\}$ edges in $G\}$ (since G has no isolated vertices)

= min size of an edge cover $F \subseteq E$ in G

$=: \rho(G) = \alpha(G)$ by König-Egerváry + Gallai Thms ∎

③ Recall for a multigraph $G=(V,E)$,

its line graph $\mathrm{line}\,G := \big(\underset{\overset{\parallel}{E}}{V_{\mathrm{line}\,G}},\ \underset{\overset{\parallel}{\{\{e,e'\}:\ e,e' \text{ are incident to some } x\in V \text{ in } G\}}}{E_{\mathrm{line}\,G}}\big)$



PROPOSITION: G bipartite $\Rightarrow$ line G is perfect.

proof: For a bipartite multigraph G,

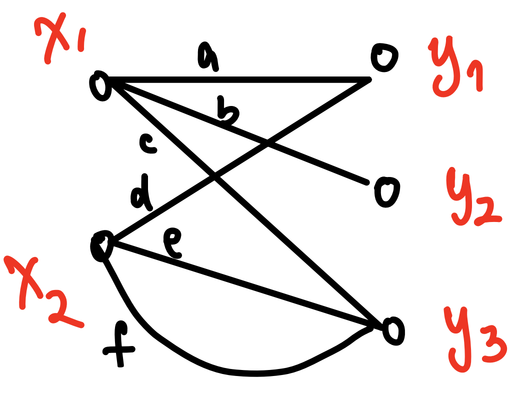
note $\omega(\mathrm{line}\,G) = \Delta(G)$

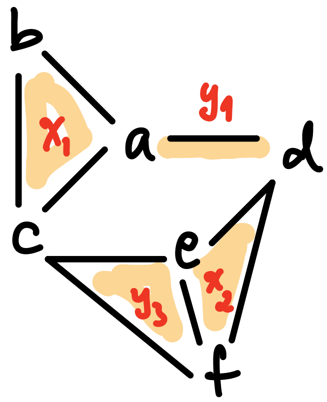
(slight subtlety: G has no triangles)

$\parallel$ ← This is König's Line Coloring Theorem!

$\chi(\mathrm{line}\,G) = \chi'(G)$

$G :=$





$\overline{G}$

Also line graphs are closed under taking vertex-induced subgraphs – they correspond to line graphs of edge-subgraphs of G ▨

④ **PROPOSITION:** G bipartite $\Rightarrow$ $\overline{\text{line}\,G}$ is perfect

proof: Note $\underbrace{\omega(\overline{\text{line}G})}_{\text{max clique size in } \overline{\text{line}G}} = \underbrace{\alpha(\text{line}\,G)}_{\text{max indep. size in line}G} = \underbrace{\nu(G)}_{\text{max matching size in } G}$

while $\chi(\overline{\text{line}G}) = \min\{k : V_{\overline{\text{line}G}} = V_1 \sqcup \cdots \sqcup V_k,\ \text{each } V_i \text{ indep. in } \overline{\text{line}G}\}$

i.e. a clique in line G

$= \min\{k : E = E_1 \sqcup \cdots \sqcup E_k,\ \text{each } E_i \text{ edge sets sharing a common incident vertex in } G\}$

again same slight subtlety: G has no triangles

$=$ min size vertex cover $W \subseteq V$ in G

$= \tau(G)$

$= \nu(G)$ by König-Egerváry again. ▨

These and other examples of families of perfect graphs closed under $G \leftrightarrow \overline{G}$ led Berge (1963) to two conjectures.

First note the odd cycles $C_5, C_7, C_9, \ldots$ are not perfect since they have $\chi(C_n) = 3 > 2 = \omega(C_n)$

CONJECTURE (Weak Perfect Graph Conjecture) G perfect $\Leftrightarrow$ $\overline{G}$ perfect

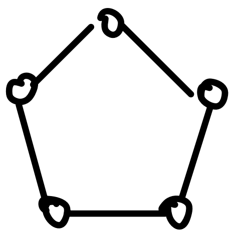
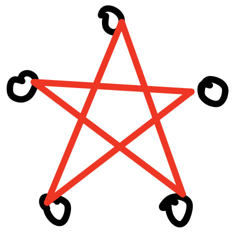
$\Uparrow$

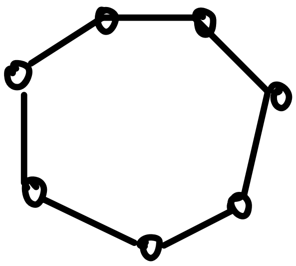
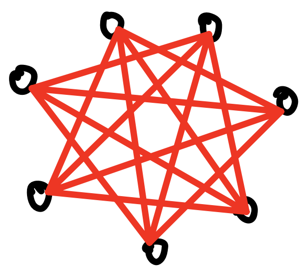
CONJECTURE

(Strong Perfect Graph Conjecture)

G perfect $\Leftrightarrow$ G has no vertex-induced subgraph $G' = G[V']$ isomorphic to C_n or $\overline{C_n}$ for $n = 5, 7, 9, \dots$ odd

i.e. no "odd holes" and no "odd anti-holes"

C_5  $\cong$  $\overline{C_5}$

C_7  $\ncong$  $\overline{C_7}$

$\vdots$ $\vdots$

Lovasz proved the Weak Perfect Graph conjecture, by proving the following stronger statement.

THEOREM (Lovasz 1972): G a simple graph is perfect $\iff$ every vertex-induced subgraph $G' = G[V']$ has

$$\underbrace{\omega(G')}_{\text{max size clique}} \cdot \underbrace{\alpha(G')}_{\text{max size indep set}} \geq |V'| \quad (*)$$

proof:

($\Rightarrow$) For any graph G, one has $\chi(G) \cdot \alpha(G) \geq |V|$ because a proper $\chi(G)$-coloring decomposes $V = V_1 \sqcup V_2 \sqcup \cdots \sqcup V_{\chi(G)}$ with $V_i \subseteq V$ indep. sets

$$\text{so } |V| = \sum_{i=1}^{\chi(G)} \underbrace{|V_i|}_{\leq \alpha(G)} \leq \chi(G) \cdot \alpha(G).$$

Hence for perfect G, one has $\underbrace{\omega(G)}_{=\chi(G)} \cdot \alpha(G) \geq |V|$,

and the same inequality is inherited by all of its vertex-induced subgraphs G', since they are also perfect.

($\Leftarrow$) (different proof by Gasparian 1996)

Suppose G is not perfect but satisfies $(*)$, and has $n := |V|$ smallest among all such examples.

We'll reach a contradiction.

We know $\chi(G) > \omega(G)$, but G' is perfect $\forall G' = G[V'] \subsetneq G$.

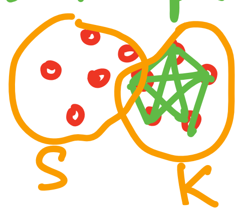
Letting $\alpha := \alpha(G)$, $\omega := \omega(G)$, we'll use linear algebra to produce the contradiction as follows.

We'll construct indep. sets $S_0, S_1, \dots, S_{\alpha\omega}$ in G

and cliques $K_0, K_1, \dots, K_{\alpha\omega}$ in G

with $|K_i \cap S_j| = \begin{cases} 0 & \text{if } i = j \\ 1 & \text{if } i \neq j \end{cases}$

[NOTE: $|K \cap S| \leq 1$ for a clique K and indep set S]



This would imply that their (0,1)- incidence matrices

$$A = \begin{array}{c} \\ K_0 \\ K_1 \\ \vdots \\ K_{\alpha\omega} \end{array} \begin{array}{c} x_1 \; x_2 \; \cdots \; x_n \\ \left[\begin{array}{c} \\ a_{ij} = \begin{cases} 1 & \text{if } x_j \in K_i \\ 0 & \text{else} \end{cases} \\ \\ \end{array} \right] \end{array} \quad \text{and} \quad B = \begin{array}{c} \\ S_0 \\ S_1 \\ \vdots \\ S_{\alpha\omega} \end{array} \begin{array}{c} x_1 \; x_2 \; \cdots \; x_n \\ \left[\begin{array}{c} \\ b_{ij} = \begin{cases} 1 & \text{if } x_j \in K_i \\ 0 & \text{else} \end{cases} \\ \\ \end{array} \right] \end{array}$$

where $V = \{x_1, x_2, \dots, x_n\}$

satisfy $(A \cdot B^T)_{ij} = |K_i \cap S_j| = \begin{cases} 0 & \text{if } i = j \\ 1 & \text{if } i \neq j \end{cases}$

That is, $AB^T = \begin{bmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & 0 \end{bmatrix} = \underbrace{\underbrace{J_{\alpha\omega+1}}_{\substack{\text{has eigenvalues} \\ (\alpha\omega+1,0,0,\ldots,0)}} - I_{\alpha\omega+1}}_{\text{has eigenvalues } (\alpha\omega,-1,-1,\ldots,-1)}$

$\Rightarrow AB^T$ is nonsingular (no zero eigenvalues)

$\Rightarrow \operatorname{rank}(AB^T) = \alpha\omega+1$

$\Rightarrow \operatorname{rank}(A), \operatorname{rank}(B) \geq \alpha\omega+1$

$\Rightarrow n \geq \alpha\omega+1$. Contradiction to $(*)$.

To construct the sets $\{S_i\}, \{K_i\}$,

first pick $S_0 = \{x_1, x_2, \ldots, x_\alpha\}$ any indep. set achieving $|S_0| = \alpha$ $(=\alpha(G))$.

Then for each $x \in S_0$, since $G-\{x\}$ is perfect and $\omega(G-\{x\}) \leq \omega(G) = \omega$, one can disjointly cover $G-\{x\}$ by $\chi(G-\{x\})$ indep sets. Assemble these to produce

$S_1, \ldots, S_\omega$ (covering $S_0 - \{x_1\}$)

$S_{\omega+1}, \ldots, S_{2\omega}$ (covering $S_0 - \{x_2\}$)

$\vdots$

$S_{(\alpha-1)\omega+1}, \ldots, S_{\alpha\omega}$ (covering $S_0 - \{x_\alpha\}$)

We then CLAIM $(**)$: $S_0, S_1, \ldots, S_{\alpha\omega}$ cover each $y \in V$ exactly α times, i.e. y lies in exactly α of them:

If $y \notin S_0$, it was covered once for each $x \in S_0$.

If $y \in S_0$, it was covered once for each $x \in S_0 - \{y\}$, once in S_0.

Next we claim that for each S_i, $i=0,1,\dots,\alpha\omega$ one has $\omega(G\setminus S_i)=\omega=\omega(G)$,

otherwise $\chi(G\setminus S_i)\underset{\substack{\uparrow \\ G\setminus S_i \text{ perfect}}}{=}\omega(G\setminus S_i)\leq\omega(G)-1$

and then $\chi(G)\leq\omega(G)$ by coloring S_i its own color;

contradiction to $\chi(G)<\omega(G)$.

Thus $\exists$ a clique K_i of size ω in $G\setminus S_i$,

i.e. $K_i\cap S_i=\emptyset$ for each $i=0,1,\dots,\alpha\omega$.

Lastly, we claim the inequality $|K_i\cap S_j|\leq 1$

is actually an equality $|K_i\cap S_j|=1$ because of CLAIM (**):

if $K_i=\{y_1,y_2,\dots,y_\omega\}$, then the α different sets S_j

containing y_1 must be chosen among $\{S_0,S_1,\dots,S_{\alpha\omega}\}\setminus\{S_i\}$

and they must all be different from those containing

$\{y_2,y_3,\dots,y_\omega\}$. This forces $|K_i\cap S_j|=\begin{cases}0 & \text{if } i=j\\ 1 & \text{if } i\neq j\end{cases}$ ▨

REMARKS:

① Berge's Strong Perfect Graph Conjecture was later

proven by Chudnovsky, Robertson, Seymour & Thomas (2002)

in a paper of about 150 pages (!)

② Grötschel, Lovasz and Schrijver (1981) showed that for perfect graphs G there is a polynomial-time algorithm to compute $\omega(G) = \chi(G)$ and find a proper $\chi(G)$-coloring, using the ellipsoid method in linear programming. However, it is not really a combinatorial algorithm.