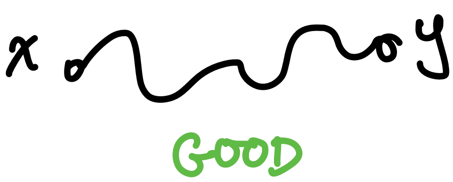
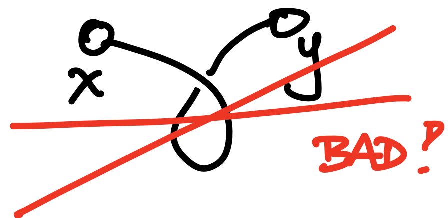
Planar graphs (Bondy-Murty Chap. 9)

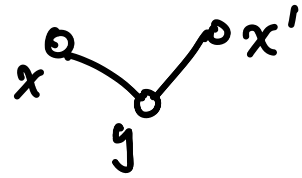
DEFINITION: A multigraph $G=(V,E)$ is planar if it can be drawn in $\mathbb{R}^2$ with

- each edge $e=\{x,y\}$ a simple Jordan curve
  ← no self-intersections

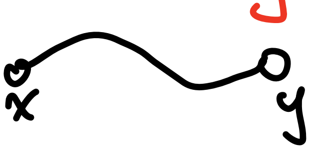




- two edges never cross or intersect at all except possibly at a common end-vertex:

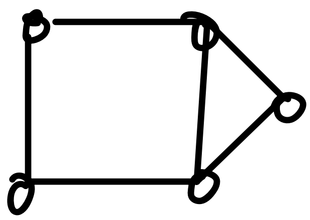


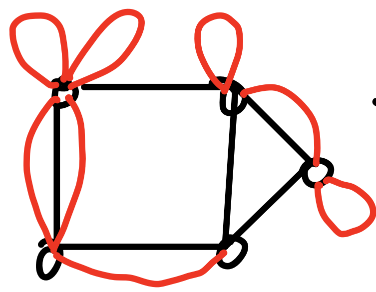
REMARK: There are some topological technicalities about the continuous embeddings $G \hookrightarrow \mathbb{R}^2$ that one can be more careful about, e.g. by insisting on the edges being embedded smoothly or piecewise-linearly



EXAMPLES:

① Adding loops or parallel edges has no effect on planarity,

e.g. $G =$  is planar

$\iff$ $\hat{G} =$  is planar

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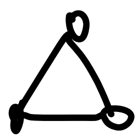
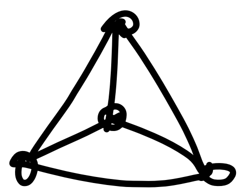
② We saw K_n is planar for $n \leq 4$

K_1

K_2

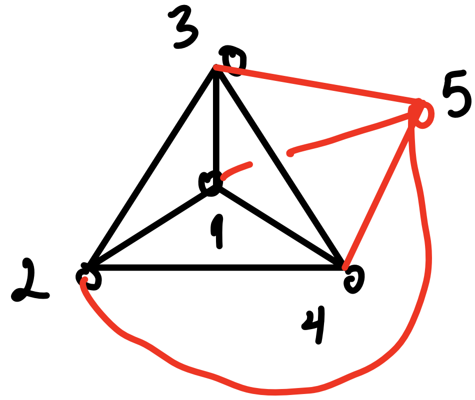


K_3



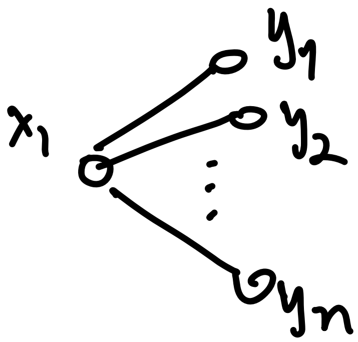
K_4

but we suspect it is non-planar for $n \geq 5$:

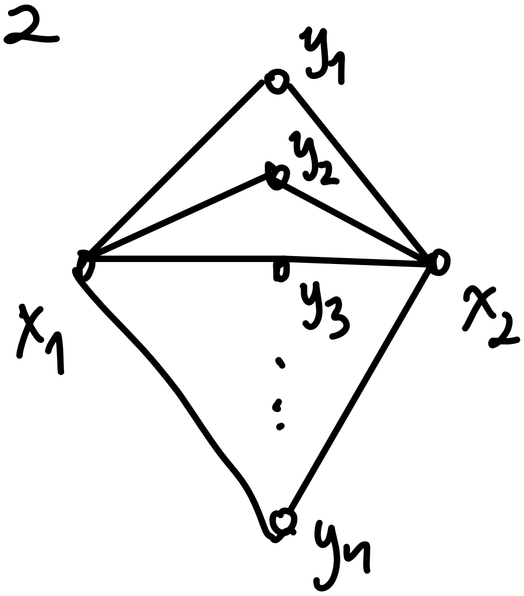


K_5

③ Similarly, we saw $K_{m,n}$ is planar if $m \leq 2$ or $n \leq 2$

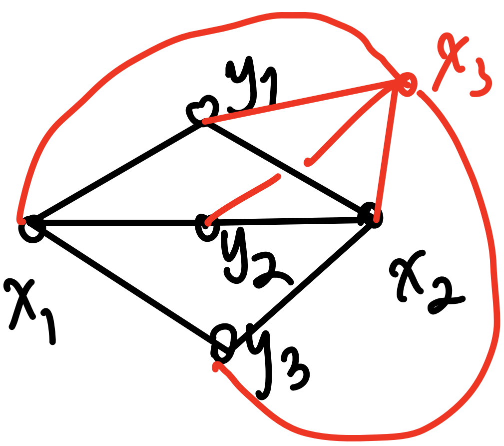


$K_{1,n}$



$K_{2,n}$

but we suspect it is non-planar for $m,n \geq 3$:

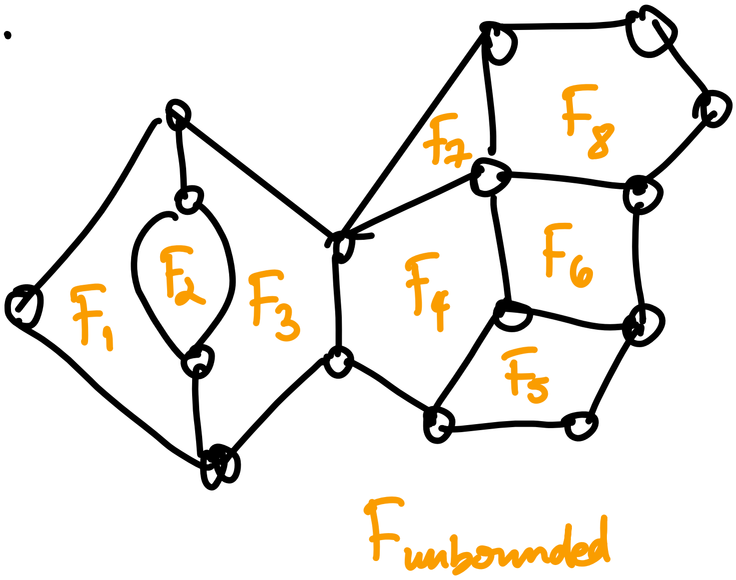


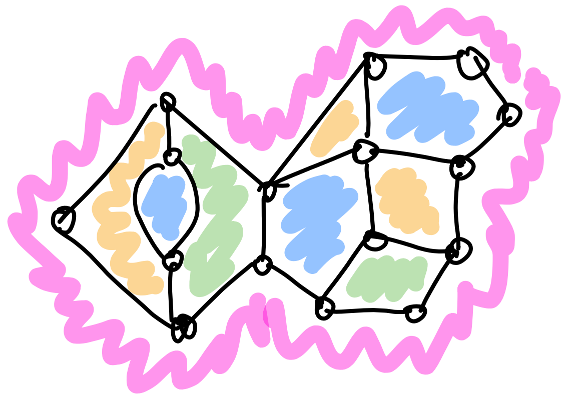
NOTE: Showing K_5 and $K_{3,3}$ are non-planar would suffice, since if $G=(V,E)$ has an edge-subgraph $G'=(V',E')$ with $V' \subseteq V$, $E' \subseteq E$, which is non-planar, then so is G.

(4) Planar maps $M = M(G)$

:= a plane embedded graph G, together with the faces/regions/countries into which it divides the plane $\mathbb{R}^2$, including one unbounded region

e.g.

$G =$ 



QUESTION: (Guthrie's "4-color problem")
1852

In a planar map $M = M(G)$, can one always color the regions with only 4 colors so that those sharing a boundary get different colors?

dual reformulation

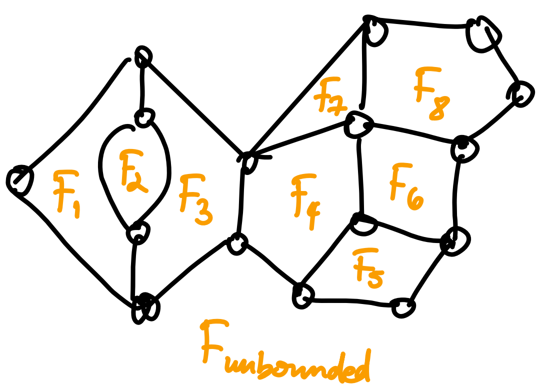
Consider the dual planar graph

$$G^* := (V^*, E^*)$$

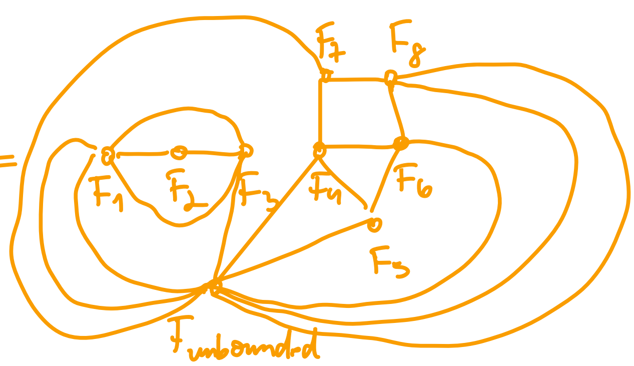
$V^* = \{\text{regions } F_1, F_2, \dots \text{ for } M(G)\}$

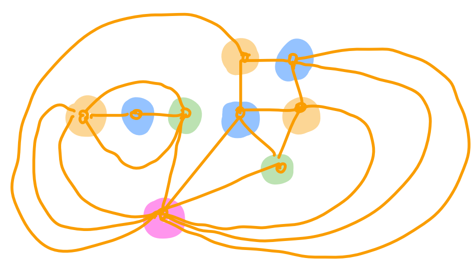
$E^* = \{\text{pairs } \{F_i, F_j\} \text{ of regions sharing a boundary}\}$

$M(G)$



$\rightsquigarrow G^* =$





EQUIVALENT QUESTION (to 4-Color Problem):

Does every planar graph G^* have $\chi(G^*) \leq 4$?

THEOREM: "4-Color Theorem"
(Appel & Haken) 1976

Yes, G planar $\Rightarrow \chi(G) \leq 4$.

However, the only currently known proofs are computer-assisted and involve case-by-case checking of related objects.

We will at least see how to prove

the 6-Color Theorem, and then
the 5-Color Theorem

pretty easily, once we have learned

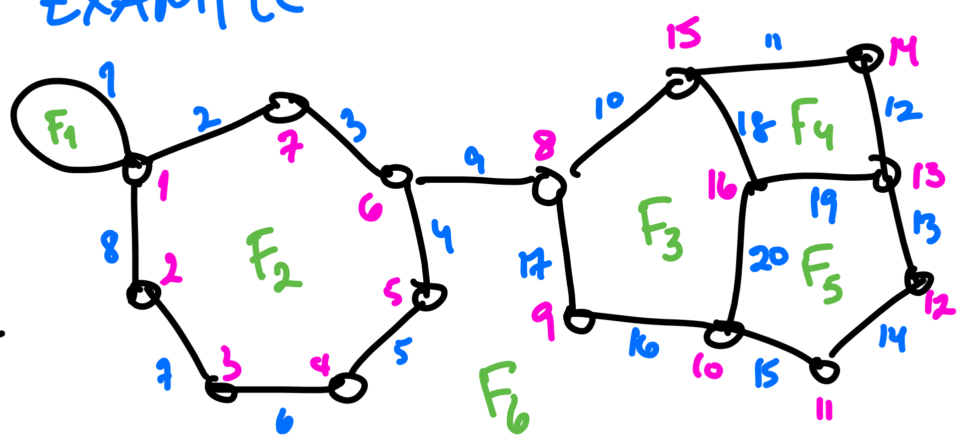
Euler's Formula.

Euler's Formula

THEOREM (Euler 1752) For any connected planar graph $G=(V,E)$ embedded in $\mathbb{R}^2$ with f faces/regions (including $F_{\text{unbounded}}$), one has

$$\underbrace{n}_{:=|V|} - \underbrace{m}_{:=|E|} + f = 2$$

EXAMPLE

$G=$ 

$n=16$

$m=20$

$f=6$

$16-20+6=2$

$n-m+f=2$

---

proof: Induct on the number f of regions.

BASE CASE: $f=1$

Then G has no cycles (else it would enclose a bounded region F, so $f\geq 2$), so G is a tree.

But then $|E|=|V|-1$,

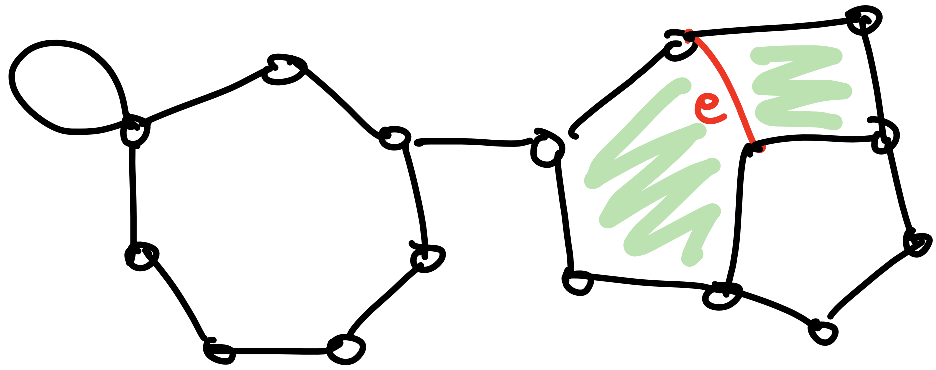
$m=n-1$

so $n-m+f$

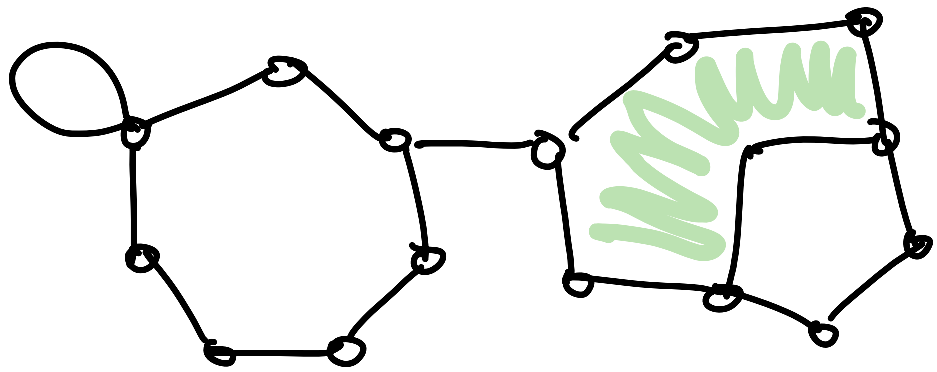
$= n-(n-1)+1 = 2$ ✓

INDUCTIVE STEP: If $f \geq 2$, then G contains a cycle, and hence $\exists$ some edge e on this cycle, whose removal does not disconnect G, but instead merges two faces

$G =$



$G \setminus e =$



Then by induction,

$$\underbrace{n(G \setminus e)}_{= n(G)} - \underbrace{m(G \setminus e)}_{= m(G) - 1} + \underbrace{f(G \setminus e)}_{= f(G) - 1} = 2$$

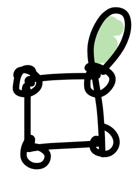
$$\Rightarrow \quad n(G) - m(G) + f(G) = 2 \qquad \blacksquare$$

COROLLARIES to Euler's formula

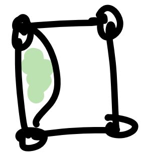
DEFINITION: The girth of a multigraph $G=(V,E)$ is the number of edges in its shortest cycle.
If G has no cycles then $\mathrm{girth}(G) := \infty$.

EXAMPLES

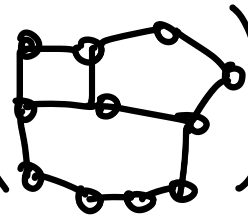
- $\mathrm{girth}(G)=1 \iff G$ has a loop



- $\mathrm{girth}(G)=2 \iff G$ is loopless, but has a parallel ed e somewhere



- $\mathrm{girth}(G) \in \{3,4,5,\ldots\} \iff G$ is simple and has at least one cycle

e.g. girth() $=4$

- G bipartite $\Rightarrow \mathrm{girth}(G)$ even (or ∞)

(why no backward implication $\Leftarrow$?)

COROLLARY 1 (to Euler's formula)

A planar connected graph $G=(V,E)$ with $\text{girth}(G) \geq g$ and at least one cycle (not a tree) has

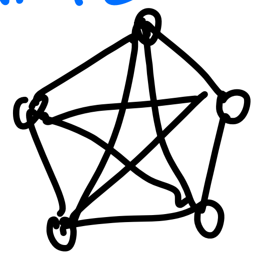
$$\underbrace{m}_{=|E|} \leq \frac{g}{g-2}(\underbrace{n}_{=|V|}-2).$$

In particular,

- simple planar connected graphs which are not trees have $m \leq \frac{3}{1}(n-2) = 3n-6$

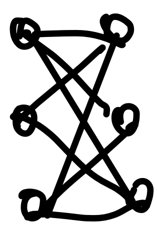
- simple planar connected bipartite graphs which are not trees have $m \leq \frac{4}{2}(n-2) = 2n-4$

EXAMPLES:



K_5 is not planar because and $\underset{\overset{\shortparallel}{10}}{m} \not\leq \underset{\overset{\shortparallel}{3\cdot5-6\,=\,9}}{3n-6}$

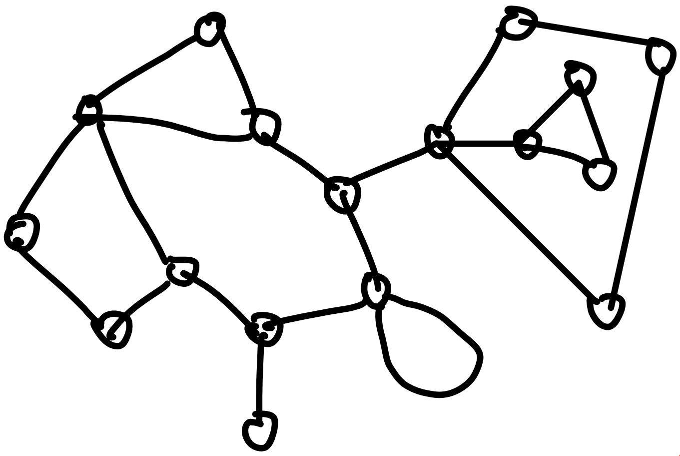
$n=5$
$m=\binom{5}{2}=10$



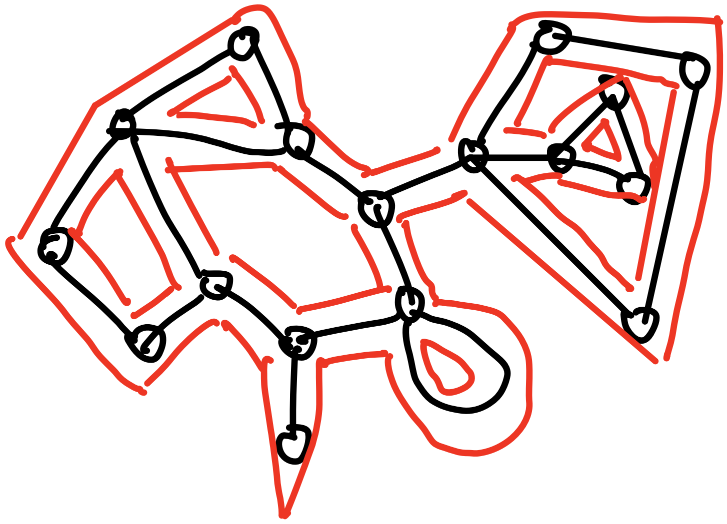
$K_{3,3}$ is not planar because and $\underset{\overset{\shortparallel}{9}}{m} \not\leq \underset{\overset{\shortparallel}{2\cdot6-4\,=\,8}}{2n-4}$

$n=6$
$m=9$

proof of COR 1: We claim that $\boxed{2m \geq gf}$ follows by counting in 2 ways the doubled red edges in the picture here,



$= G$



$$2m = \sum_{e \in E} 2 = \left| \{ (e,F) : e \in E,\ F \text{ a face},\ e \text{ bounds } F \} \right| = \sum_{\text{faces } F} \underbrace{\left| \{ \text{edges } e \text{ bounding } F \} \right|}_{\geq g} \geq fg$$

Therefore $f \leq \frac{2m}{g}$, so then Euler's formula gives

$$n - m + f = 2$$

$$n - m + \frac{2m}{g} \geq 2$$

$$n - 2 \geq m\left(1 - \frac{2}{g}\right) = m \cdot \frac{g-2}{g}$$

$$\frac{g}{g-2}(n-2) \geq m. \quad \blacksquare$$

"6-color Theorem"

COROLLARY 2: Simple planar graphs G always have a vertex of degree ≤ 5, and hence all (loopless) planar graphs can be properly vertex-6-colored, via a greedy algorithm.

proof: WLOG our simple planar graph G is connected, and has a cycle (else it's a tree, so it has a vertex of degree 1).

So it has $m \leq 3n-6$

$2m \leq 6n-12$, and

average degree $\frac{1}{n} \cdot \sum_{x \in V} d_G(x) = \frac{2m}{n} \leq \frac{6n-12}{n} = 6 - \frac{12}{n} < 6$.

Hence G has some $x \in V$ with $d_G(x) \leq 5$.

If one arranges this vertex x to come last in a greedy 5-coloring of $G - \{x\}$, which exists by induction on $n = |V|$, it won't need more than 5 colors for x.

For a loopless planar graph, one can consider its underlying simple graph. ∎

COROLLARY 3: Loopless planar graphs can be properly vertex-5-colored.

"5-color Theorem"

(Kempe 1879)

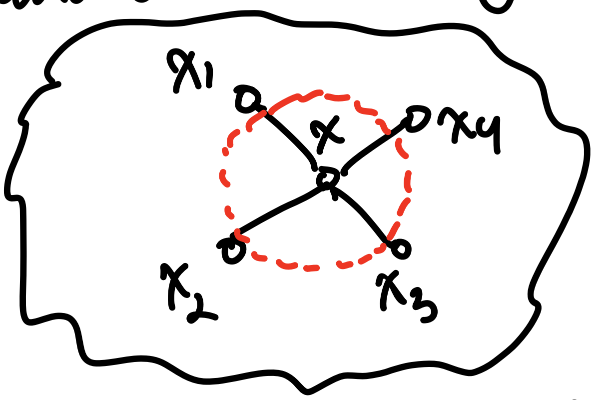
proof: Induct on $n = |V|$.

BASE CASE: $n \le 5$. No problem!

INDUCTIVE STEP: $n \ge 6$.

CASE 1: $\exists$ a vertex $x_0 \in V$ with $\deg_G(x_0) \le 4$.

Then properly 5-color $G - \{x_0\}$ by induction, and one has enough colors for x_0.



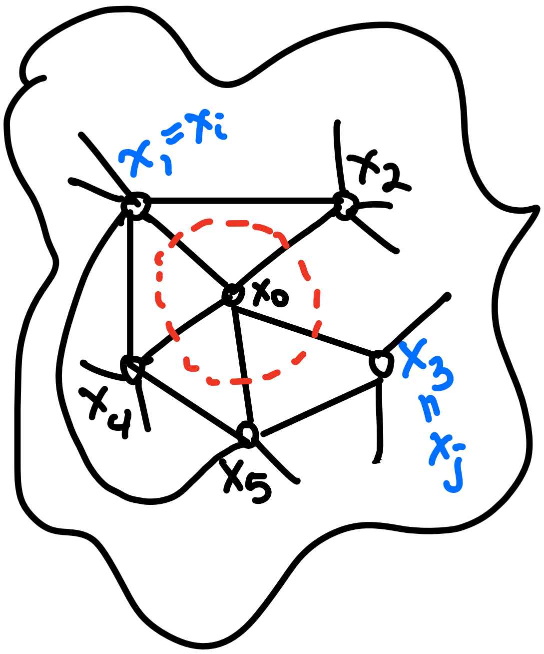
CASE 2: $\exists$ a vertex $x_0 \in V$ with $\deg_G(x_0) = 5$.

Then some pair $\{x_i, x_j\}$ of its neighbors $\{x_1, \ldots, x_5\}$ do not form an edge in G, else they would form a K_5 in G, making G non-planar.

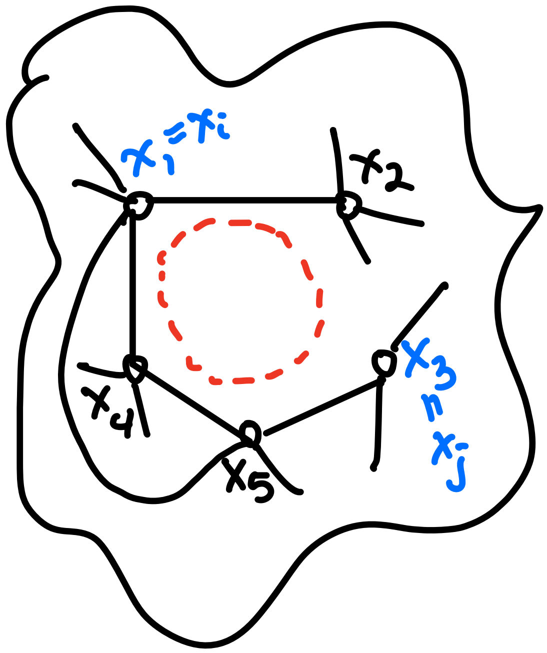
Perform this two-step construction

$$G \rightsquigarrow G - \{x_0\} \rightsquigarrow \hat{G} := \left(G - \{x_0\}\right) / _{x_i = x_j}$$

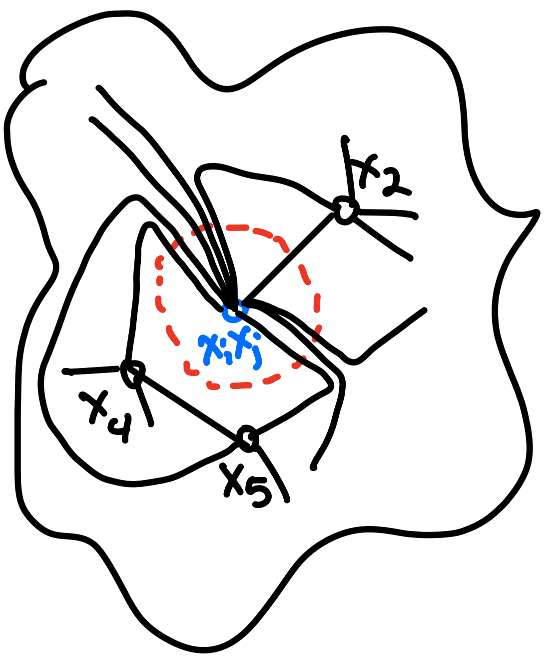
and we claim that $\hat{G}$ is still planar:



G



$G - \{x_0\}$



$$\hat{G} = \left(G - \{x_0\}\right) / {x_i = x_j}$$

By induction, $\hat{G}$ has a proper 5-coloring, which gives a proper 5-coloring of $G - \{x_0\}$ having same color on x_i, x_j, which then extends to a 5-coloring of G on x_0 ∎

It was a major achievement when Appel & Haken 1976 were able to use enough theory (some of it called discharging) to reduce the proof to a check of finitely many reducible configurations, and then check them via computer, proving...

THEOREM "4-color Theorem": Loopless planar graphs G can be properly vertex-4-colored, i.e. $\chi(G) \leq 4$.

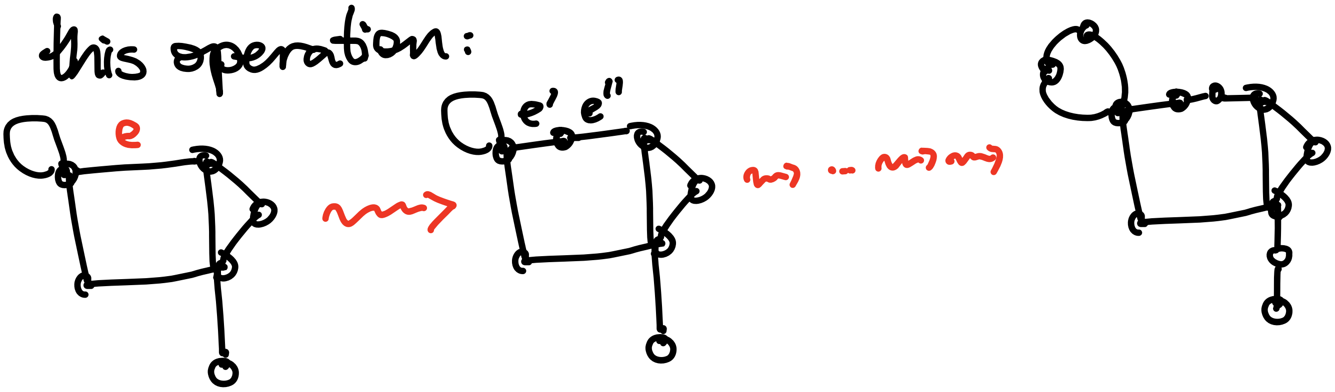
See the Wikipedia page on 4-color theorem for a lot of history and discussion.

ACTIVE EPISTEMOLOGY In which of these scenarios would you say you know a theorem is true?

- You proved it, via induction.
- You proved it, avoiding induction.
- You read a proof once, but have forgotten it.
- There is a published proof of 10 pages
  100
  1000
- There is a published proof needing a computer to check cases.
- There is a published proof checked via Lean or Rocq

Kuratowski's Theorem & graph minors
($\S$9.4, 9.5, 8.3)

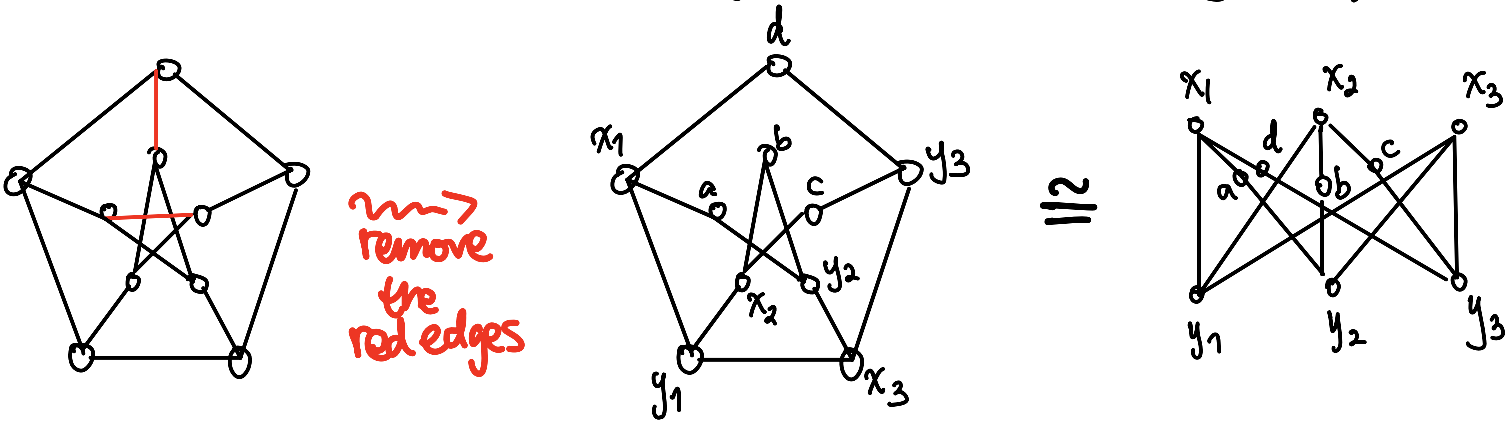
DEFINITION: Given a multigraph G, say G' is a subdivision of G if it is obtained by iterating this operation:



(easy) PROPOSITION: When G' is a subdivision of G, then G is planar $\Longleftrightarrow$ G' is planar

Hence planar graphs cannot contain edge-subgraphs isomorphic to subdivisions of K_5 or $K_{3,3}$.

EXAMPLE Petersen graph is not planar, because it has such an edge-subgraph subdividing $K_{3,3}$:



THEOREM (Kuratowski 1930)

G is planar $\iff$ it contains no edge-subgraph isomorphic to a subdivision of K_5 or $K_{3,3}$

($\underset{\text{easy}}{\Longrightarrow}$)

The proof is not so hard, but takes work - see Bondy & Murty §9.4, 9.5

An interesting variant uses this notion.

DEFINITION: Say G' is a **minor** of G if it can be obtained by a sequence of deletions and contractions of edges of G and deletions of vertices.

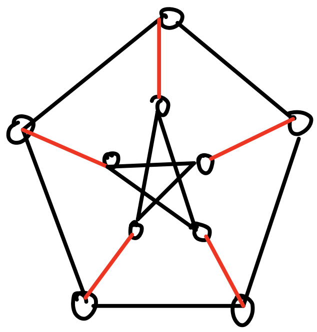
Note deleting edges, vertices and contracting edges all preserve planarity:

THEOREM (Wagner 1937)

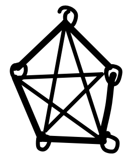
G is planar $\iff$ G has no minor $G' \cong K_5$ or $K_{3,3}$

($\Rightarrow$) is again easy, ($\Leftarrow$) is easy if one assumes Kuratowski's Theorem since any edge subdivision of G' can be contracted to G'.

EXAMPLE Petersen graph is not planar also because it has a minor $G' \cong K_5$:



contract the 5 red edges



$G' \cong K_5$

REMARK: Hadwiger (1943) posed the following:

CONJECTURE:

$$\chi(G) \geq k \iff G \text{ has a minor } G' \cong K_k$$

$\Downarrow$ (since planar graphs cannot have $K_5, K_6, \dots$ as minors)

4-COLOR THEOREM

Hadwiger's Conjecture is considered extremely hard.

Hadwiger also formulated/conjectured the following:

GRAPH MINORS THEOREM: (Robertson-Seymour 1983-2004) 500 page proof! Every graph property closed under taking minors is characterized by a finite list of excluded minors $\{G_1, G_2, \dots, G_r\}$

EXAMPLES:

① G is planar $\Longleftrightarrow$ G excludes minors $\{K_5, K_{3,3}\}$

Wagner's Theorem

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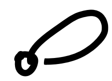
② **ACTIVE LEARNING:** Prove the following:

G is a forest (acyclic) $\Longleftrightarrow$ G excludes minor $\{C_1\}$



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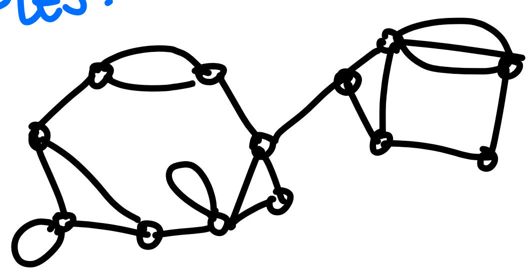
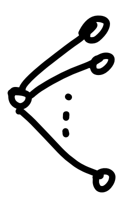
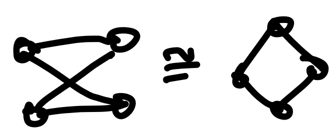
③ **DEFINITION:** A multigraph $G=(V,E)$ is called outerplanar if it has a plane embedding with every vertex $x \in V$ incident to the unbounded face.

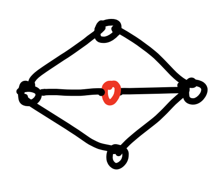
EXAMPLES:

-  is outerplanar

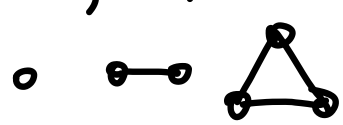
- $K_{1,n}$ and $K_{2,2}$ are outerplanar, but $K_{2,3}$ is not

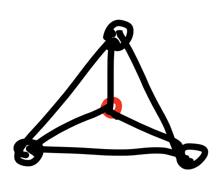




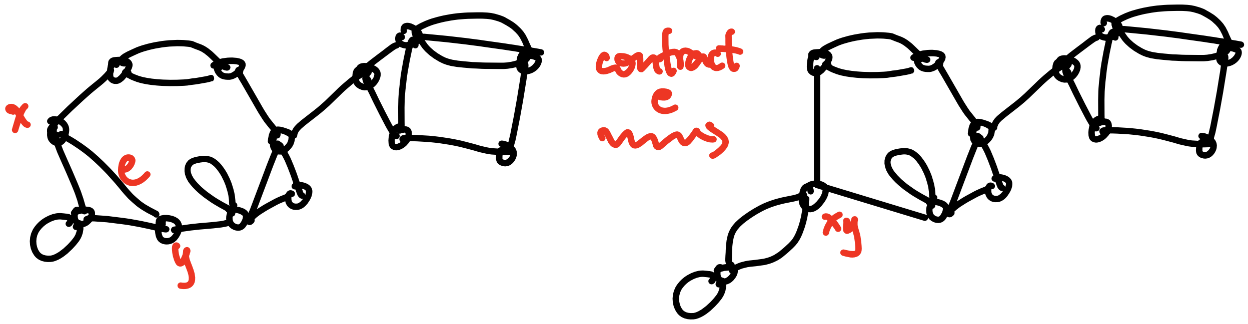


- K_1, K_2, K_3 are outerplanar, but K_4 is not





- Outerplanarity is closed under taking minors



THEOREM: (Chartrand & Harary 1967)

G is outerplanar

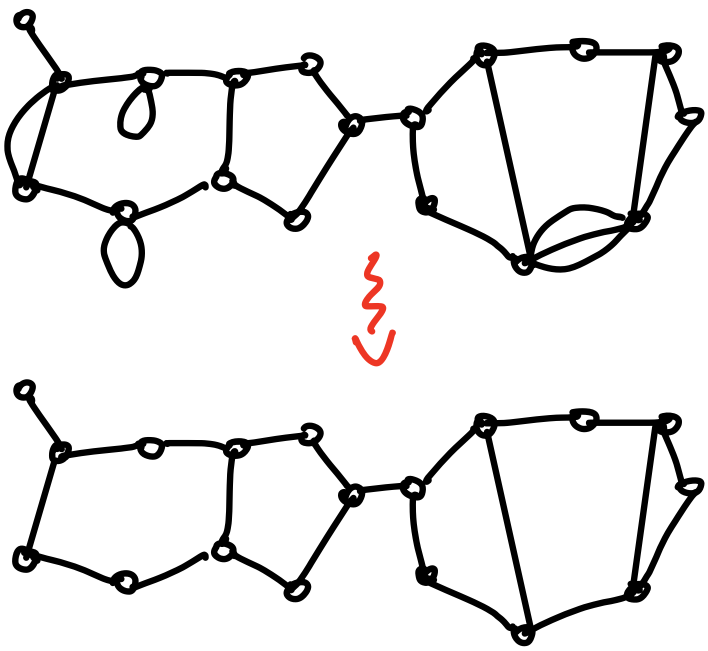
$\iff$ G excludes minors $\{K_{2,3}, K_4\}$

Consistent with Hadwiger's Conjecture, one has...

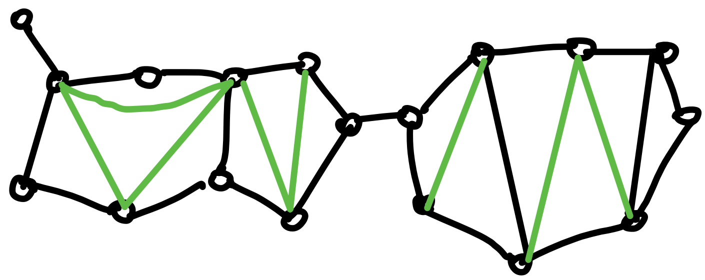
PROPOSITION: G outerplanar $\Rightarrow \chi(G) \leq 3$.

proof: sketch WLOG G is simple and outerplanar.

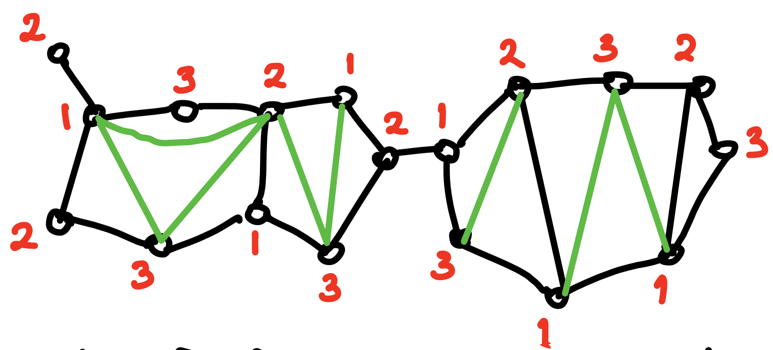
G =



Subdivide its longer cycles into triangular cycles, introducing no new vertices.



Then inside each 2-connected component, a proper 3-coloring is now unique, once you've 3-colored one of its triangles:



This restricts to a proper 3-coloring for the original graph G. ∎