Math 5707 Graph theory
Spring 2023, Vic Reiner
Midterm exam 1- Due Wednesday, March 1 by midnight at the class Canvas site

Instructions: This is an open book, open library, open notes, open web, take-home exam, but you are *not* allowed to collaborate. The instructor is the only human source you are allowed to consult.

1.(20 points total) **True** or **False**. To receive full credit, assertions which are true must be proven, and those that are false must have a counterexample exhibited.

(a) (5 points) There exists a **simple** graph (no loops, no parallel edges) having degree sequence $(d_1, \ldots, d_n) = (6, 6, 6, 6, 5, 3, 3, 1)$.

(b) (5 points) For $a, b \geq 1$, the complete bipartite graph $K_{a,b}$ contains an Euler circuit (closed Euler tour) if and only if a, b are both even.

(c) (5 points) For $a, b \geq 2$, the complete bipartite graph $K_{a,b}$ contains a Hamilton circuit (closed Hamilton tour) if and only if $a = b$.

(d) (5 points) Any simple graph contains either an Euler circuit or Hamilton circuit, or both.

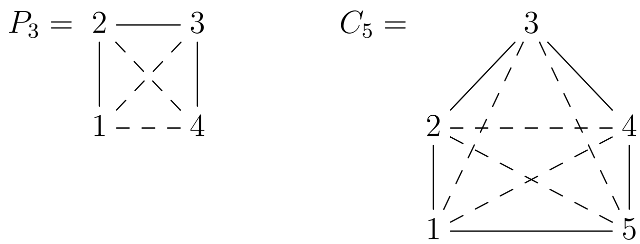
2.(20 points) Given a multigraph $G = (V, E)$ and any edge-cost function $c : E \to \mathbb{R}_{\geq 0}$, let T be a minimum cost spanning tree, that is, a spanning tree $T \subseteq E$ that achieves the smallest value of $c(T) := \sum_{e \in T} c(e)$.

Prove or disprove (via counterexample): For every pair of vertices x, y in V, the unique path P from x to y within T will achieve the minimum cost $c(P) = \sum_{e \in P} c(e)$ among *all* paths from x to y within G.

3. (20 points) Show that a tree with at least one edge that has no vertices of degree two will have more leaf vertices (that is, degree one vertices) than non-leaf vertices, and in fact, at least *two more* leaves than non-leaves.

4. (20 points total) Recall that for a simple graph $G = (V, E)$, its *complement graph* $\bar{G}$ is the simple graph on the same vertex set V, but with the complementary set of edges. That is, $\{x, y\}$ is an edge of $\bar{G}$ if and only if $\{x, y\} \notin E$.

Say G is *self-complementary* if $\bar{G}$ is isomorphic to G. For example, the path P_3 with 3 edges is self-complementary, as is the 5-cycle C_5 (edges of $\bar{G}$ are shown dashed):



We proved on Homework 1, in Bondy and Murty's Exercise 1.2 #11, that a self-complementary graph must have $n = |V|$ either congruent to 0 or 1 modulo 4, that is, either n is divisible by 4 or n has remainder 1 on division by 4.

(a) (10 points) Prove that for any n divisible by 4 there exists a self-complementary graph having n vertices.
(Hint: break the vertex set into 4 equal size groups and then use the path P_3 as a guideline for how to connect between groups).

(b) (10 points) Prove that for any n having remainder 1 on division by 4 there exists a self-complementary graph having n vertices.
(Hint: Figure out how to add one more vertex to your construction from part (a).)

5. (20 points) Let $G = (V, E)$ be a simple graph with $\deg_G(v) \geq 2$ for all vertices v in V. Prove that there exists a **connected** simple graph H having the same (weakly decreasing reoredered) degree sequence $\mathbf{d}(H) = (d_1 \geq d_2 \geq \cdots \geq d_n) = \mathbf{d}(G)$ as the one for G.