Math 5707 Graph theory
Spring 2025, Vic Reiner
Final exam
Due Wednesday, May 7 by midnight
via Gradescope at the class Canvas site

Instructions: This is an open book, open notes, open web, takehome exam, but you may not collaborate. You must clearly indicate any such sources used, including any AI sources, such as ChatGPT. The instructor is the only human source whom you are allowed to consult.

1.(20 points) A *tournament* is an orientation Ω of the edges of the complete graph K_n for some $n \geq 2$. Prove every tournament contains a *directed Hamiltonian path*, that is, an ordering of the vertices $x_1, x_2, \ldots, x_n$ so that Ω contains all of these arcs:

$$x_1 \to x_2 \to \cdots \to x_{n-1} \to x_n.$$

2.(20 points) Note that one can disjointly decompose the edges $E(K_4)$ of the complete graph K_4 into the edge sets of 2-edge paths P_1, P_2, P_3:

$$E(K_4) = \{12, 13, 14, 24, 23, 34\} = \underbrace{\{12, 13\}}_{E(P_1)} \sqcup \underbrace{\{14, 24\}}_{E(P_2)} \sqcup \underbrace{\{23, 34\}}_{E(P_3)}.$$

Prove that the edges $E(K_n)$ of the complete graph K_n for $n \geq 1$ can be similarly decomposed

$$E(K_n) = E(P_1) \sqcup E(P_2) \sqcup \cdots \sqcup E(P_r)$$

into a disjoint union of edge sets of 2-edge paths P_i if and only if n is congruent to 0 or 1 modulo 4, that is, n either has remainder 0 or 1 upon division by 4.

3.(20 points) Show that every multigraph $G = (V, E)$ has an orientation Ω in which

$$|\mathrm{indeg}_\Omega(x) - \mathrm{outdeg}_\Omega(x)| \leq 1$$

for all x in V.

4.(20 points total) Recall that for a multigraph $G = (V, E)$, a choice of an orientation Ω for each of its edges to form a directed graph is called *acyclic* if it contains no directed cycles, and called *totally cyclic* if every directed arc in Ω participates in at least one directed cycle.

(a) (10 points) **True or False?** (with proof required)
The orientation Ω of the undirected multigraph $G = (V, E)$ is totally cyclic if and only if the following condition holds: for every x, y in V lying in the same connected component of G, there exists within Ω both a directed path from x to y and a directed path from y to x.

(b) (5 points) **True or False?** (with proof required)
Every loopless multigraph G has at least one acyclic orientation.

(c) (5 points) **True or False?** (with proof required)
Every multigraph G containing no cut-edges has at least one totally cyclic orientation.

5. (20 points total) For a multigraph $G = (V, E)$, let $TC(G)$ denote the set of all *totally cyclic orientations* Ω of G, and let $\mathrm{tc}(G) := |TC(G)|$ be its cardinality.

(a) (5 points) Prove that $\mathrm{tc}(G) = 0$ if G contains a cut edge.

(b) (15 points) Prove that $\mathrm{tc}(G) = \mathrm{tc}(G \backslash e) + \mathrm{tc}(G/e)$ if e is not a cut-edge of G.

[Hint for (b): similarly to part of our proof in lecture relating *acyclic* orientations to chromatic polynomials, try to define a map $f : TC(G) \to TC(G/e)$ that sends Ω to its *contraction* $\hat{\Omega} := \Omega/e$, and examine the cardinalities of the pre-images $|f^{-1}(\hat{\Omega})|$ in various cases.]