**Math 5707 Graph theory**
**Spring 2025, Vic Reiner**
**Midterm exam 1**
**Due Wednesday, March 5 by midnight at the class Canvas site**

**Instructions:** This is an open book, open notes, open web, takehome exam, but you may not collaborate. You must clearly indicate any such sources used, including any AI sources, such as ChatGPT. The instructor is the only human source whom you are allowed to consult.

1.(25 points total)

(a) (5 points) Draw the unique tree $T$ whose Prüfer code is $(3, 8, 1, 1, 6, 8)$.

(b) (10 points) How many trees $T$ on vertex set $V = \{1, 2, \ldots, 13, 14\}$ have degree sequence

$$\mathbf{d}(T) = (4, 4, 2, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1),$$

that is, $\deg_T(1) = \deg_T(2) = 4, \deg_T(3) = \deg_T(4) = 2$ and $\deg_T(x) = 1$ for $x = 5, 6, \ldots, 14$?

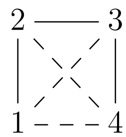
(c) (10 points) How many trees $T$ on vertex set $V = \{1, 2, \ldots, 15, 16\}$ have degree sequence

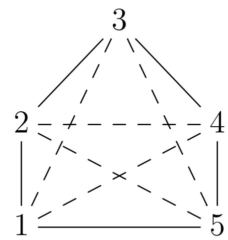
$$\mathbf{d}(T) = (5, 4, 4, 3, 2, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)?$$

2.(25 points) For $G = (V, E)$ a connected simple (undirected) graph $G = (V, E)$ *with* $|E|$ *even*, show that there is at least one orientation of its edges into directed arcs creating a digraph $D = (V, A)$ where every vertex $x$ has even outdegree. That is, construct $D = (V, A)$ having $G$ as its underlying undirected graph and $\text{outdeg}_D(x)$ *even for all* $x$ *in* $V$.

3. (25 points total) Recall that for a simple graph $G = (V, E)$, its *complement graph* $\bar{G}$ is the simple graph on the same vertex set $V$, but with the complementary set of edges. That is, $\{x, y\}$ is an edge of $\bar{G}$ if and only if $\{x, y\} \notin E$.

Say $G$ is *self-complementary* if $\bar{G}$ is isomorphic to $G$. For example, the path $P_3$ with 3 edges is self-complementary, as is the 5-cycle $C_5$ (edges of $\bar{G}$ are shown dashed):

$P_3 =$ 


$C_5 =$ 


We proved on Homework 1, in Bondy and Murty's Exercise 1.2 #11, that a self-complementary graph must have $n = |V|$ either congruent to 0 or 1 modulo 4, that is, either $n$ is divisible by 4 or $n$ has remainder 1 on division by 4.

(a) (15 points) Prove that for any $n$ divisible by 4 there exists a self-complementary graph having $n$ vertices.

(Hint: break the vertex set into 4 equal size groups and then use the path $P_3$ as a guideline for how to connect between groups).

(b) (10 points) Prove that for any $n$ having remainder 1 on division by 4 there exists a self-complementary graph having $n$ vertices.

(Hint: Figure out how to add one more vertex to your construction from part (a).)

4. (25 points) Let $G = (V, E)$ be a simple graph with $\deg_G(v) \geq 2$ for all vertices $v$ in $V$. Prove that there exists a **connected** simple graph $H$ having the same (weakly decreasing re-ordered) degree sequence $\mathbf{d}(H) = (d_1 \geq d_2 \geq \cdots \geq d_n) = \mathbf{d}(G)$ as the one for $G$.