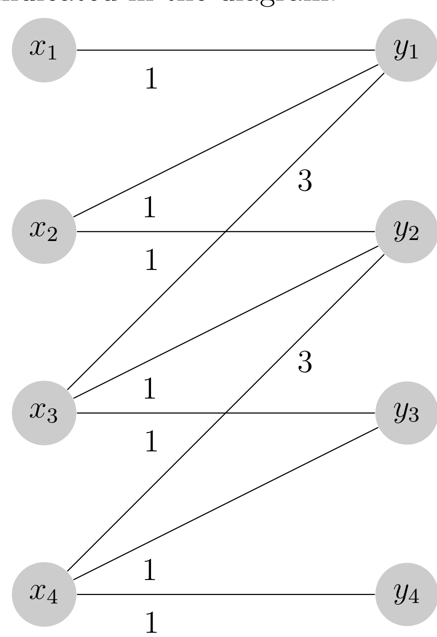
Math 5707 Graph theory
Spring 2025, Vic Reiner
Midterm exam 2
Due Wednesday, April 16 by midnight at the class Canvas site

Instructions: This is an open book, open notes, open web, takehome exam, but you may not collaborate. You must clearly indicate any such sources used, including any AI sources, such as ChatGPT. The instructor is the only human source whom you are allowed to consult.

1.(25 points) Let $G = (V, E)$ be a simple graph with $|V| = n$, and $x \in V$.

If G is k-vertex-connected for some $k \geq 1$, prove that there exists an ordering $x_1, x_2, \ldots, x_{n-1}$ of $V \setminus \{x\}$ such that $G - \{x_1, x_2, \ldots, x_i\}$ is at least k_i-vertex-connected for $i = 1, 2, \ldots, n-1$, where $k_i := \max\{k - i, 1\}$.

2.(25 points total) Consider the bipartite graph $G = (X \sqcup Y, E)$ shown below, with edge weights $w : E \to \mathbb{R}_{\geq 0}$ as indicated in the diagram.



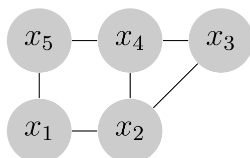
(a) (15 points) Find (with proof) a matching M of maximum weight $w(M) := \sum_{e \in M} w(e)$ among all matchings

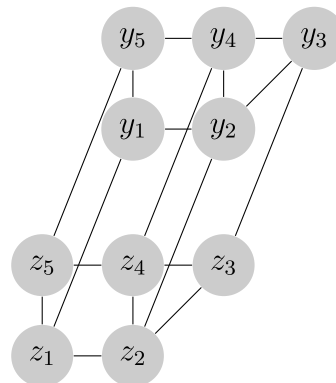
(b) (10 points) Find (with proof) a perfect matching M of maximum weight $w(M)$ among all perfect matchings.

3. (25 points) For a simple graph $G = (V, E)$ with $V = \{x_1, \ldots, x_n\}$, define its *prism graph* $P(G) = (V_{P(G)}, E_{P(G)})$ by

$$\begin{aligned} V_{P(G)}) &:= \{y_1, \ldots, y_n, z_1, \ldots, z_n\}, \\ E_{P(G)} &:= \{\{y_i, y_j\}, \{z_i, z_j\} : \text{ for each edge } e = \{x_i, x_j\} \in E\} \\ &\quad \sqcup \{\{y_i, z_i\} : i = 1, 2, \ldots, n\}. \end{aligned}$$

True or False (with proof justifying your answer required):
If $|E| \geq 1$, then $\chi(P(G)) = \chi(G)$.





4. (25 points total) Consider three simple graphs on the same vertex set V

$$F = (V, E_F),$$
$$G = (V, E_G),$$
$$H = (V, E_H).$$

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(a) (15 points) Prove that that if $E_F = E_G \sqcup E_H$, then

$$\chi(F) \leq \chi(G)\chi(H).$$

(b) (10 points) **True or False** (with proof justifying your answer required):
There exist such graphs F, G, H as in (a) with $|E_G|, |E_H| \geq 1$ and

$$\chi(F) = \chi(G)\chi(H).$$