Math 5711 Combinatorial optimization
Spring 2004, Vic Reiner
Midterm exam 2- Due Wednesday April 7, in class

Instructions: This is an open book, open library, open notes, open web, take-home exam, but you are *not* allowed to collaborate. The instructor is the only human source you are allowed to consult.

1. (20 points) Schrijver's Problem 2.25 on p. 27. He asserts the equality of a max and a min, but forgot to ask you to prove it. Prove it.

2. (20 points total) Consider the following LP problem as primal:

$$\text{maximize } x_1 + 2x_2 + 3x_3$$

$$\text{subject to} \quad \begin{array}{cl} x_1 + x_2 + x_3 & \leq 1 \\ x_1, x_2, x_3 & \geq 0 \end{array}$$

(a) (5 points) Write down the dual LP.

(b) (8 points) Disprove the assertion that $x^* = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is the optimal solution using complementary slackness. Do *not* use simplex method (or any other method) to solve the primal or dual LP; only an argument via complementary slackness will receive credit.

(c) (7 points) Prove that $x^* = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is the optimal solution using complementary slackness. Again, only an argument via complementary slackness will receive credit.

3. (15 points) In lecture, the matrix game with payoff matrix (for the row player) given by

$$\begin{bmatrix} -6 & +55 \\ +11 & -65 \\ -26 & +75 \end{bmatrix}$$

was claimed to have value $v = \frac{215}{137}$ for the row player, with row player's optimal mixed strategy given by the probabilities $x^* = \begin{bmatrix} \frac{76}{137} \\ \frac{61}{137} \\ 0 \end{bmatrix}$ and the column player's by $y^* = \begin{bmatrix} \frac{120}{137} \\ \frac{17}{137} \end{bmatrix}$.

Use LP duality to show that these claims are true, by checking certain inequalities and equations are satisfied. Explain why these checks suffice. Do *not* use simplex method (or any other method) to solve any LP's.

4. (a) (10 points) Schrijver's Problem 3.18 on page 35.
(b) (5 points) Prove (rigorously!) that the assignment in your answer to part (a) is optimal.

5. (15 points total) Recall that a graph $G = (V, E)$ is *bipartite* if one can decompose $V = U \sqcup W$ as a disjoint union in such a way that every edge $e \in E$ has the form $e = \{u, w\}$ for some $u \in U, w \in W$.

An *odd cycle* in $G = (V, E)$ is a sequence of vertices $v_1, v_2, \ldots, v_k \in V$ with k odd such that $\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{k-1}, v_k\}$ and $\{v_k, v_1\}$ are all edges in E.
(a) (7 points) Prove that a bipartite graph G has no odd cycles.
(b) (8 points) Prove that a graph $G = (V, E)$ is bipartite *if and only if* it contains no odd cycles.

6. (15 points) Schrijver's Problem 3.12 on page 33.