Math 5711 Combinatorial optimization
Spring 2004, Vic Reiner
Final exam - Due Wednesday May 3, in my Vincent Hall 105 mailbox by 4pm.

Instructions: This is an open book, open library, open notes, open web, take-home exam, but you are *not* allowed to collaborate. The instructor is the only human source you are allowed to consult.

1. (20 points total) Consider the following LP problem as primal:

$$\begin{array}{ll} & \text{maximize } 3x_1 + 2x_2 + x_3 \\ & \begin{array}{rl} x_1 + x_2 + x_3 & \geq 1 \\ \text{subject to} \qquad x_1 + x_2 + x_3 & \leq 2 \\ x_1, x_2, x_3 & \geq 0 \end{array} \end{array}$$

(a) (5 points) Use the first phase of two-phase simplex method to find a feasible dictionary for this LP problem. Show every dictionary along the way that you use (but omit algebra steps if you like).

(b) (5 points) Solve the LP via simplex method, beginning with the feasible dictionary from (a). Again, show every dictionary along the way.

(c) (5 points) Write down the dual LP in Chvátal's standard form.

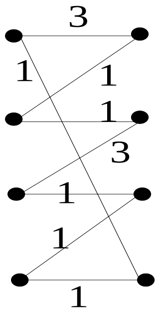
(d) (5 points) Exhibit a solution to the dual LP by any method, but explain which method you used.

2. (15 points total)

(a) (5 points) Exhibit an LP which is infeasible, but whose dual LP is feasible.

(b) (5 points) Exhibit an LP which is feasible, but whose dual LP is infeasible.

(a) (5 points) Exhibit an LP which is infeasible and whose dual LP is also infeasible.



3. (15 points) Apply Kuhn's Hungarian algorithm to find a maximum weight matching in the bipartite graph with edge weights shown above. Show each step in the algorithm.

4. (20 points) Schrijver's Problem 4.8(i) on page 50. Show all of the augmenting flow steps, and also in the end, exhibit a cut which shows your flow achieves the maximum value.

5. (15 points) In doing the National Residency Matching Program, assume that student S ranks residency program r highest on their list, and conversely that r ranks student S highest on its list. Show that, as one might expect, the Gale-Shapley algorithm will match S with r, regardless of any of the other preferences.

6. (15 points) In application 1.7 on page 21, Schrijver discusses the notion of the reliability $r_G(u, v)$ of two vertices u, v in a graph $G = (V, E)$ with given edge strengths $s : E \to \mathbb{R}$. He defines the reliability of a $u - v$ path to be the *minimum* strength $s(e)$ of all edges e on the path, and then $r_G(u, v)$ is the *maximum* reliablity of all the $u - v$ paths in G.

Prove his assertion in equation (25) on that page, namely that the reliability $r_G(u, v)$ is the same as the reliability $r_T(u, v)$ where T is any *maximum* weight spanning tree in G using the function s as the edge weights.
(Hint: circuit exchange!)