Math 8201 Graduate abstract algebra- Fall 2013, Vic Reiner
Midterm exam 1- Due Wednesday October 16, in class

Instructions: This is an open book, library, notes, web, take-home exam, but you are *not* to collaborate. The instructor is the only human source you are allowed to consult. Indicate outside sources used.

1. (15 points total; 5 points each part)
Inside $G := GL_n(\mathbb{Z}/17\mathbb{Z})$, consider the subset H consisting of those matrices A having $\det A \in \{\pm\bar{1}, \pm\bar{4}\}$.
(a) Show H is a normal subgroup of G.
(b) Identify the group G/H up to isomorphism.
(c) Compute $|H|$ as a function of n.

2. (15 points total) Let H, K be two subgroups and g any element, in a finite group G.
(a) (5 points) Prove

$$|g^{-1}Hg \cap K| = |H \cap gKg^{-1}|$$

(b) (10 points) Prove

$$|HgK| = \frac{|H||K|}{|g^{-1}Hg \cap K|} \qquad \left(= \frac{|H||K|}{|H \cap gKg^{-1}|} \text{ by part (a)} \right).$$

3. (30 point total; 5 points each part) Prove or disprove:
- (i) $D_{12} \cong \mathbb{Z}/2\mathbb{Z} \times D_6$ as groups.
- (ii) $D_{16} \cong \mathbb{Z}/2\mathbb{Z} \times D_8$ as groups.
- (iii) $\mathbb{R}^\times \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{R}^+$ as groups.
- (iv) For g, h in a finite group, the order of the product gh divides the product of the orders of g and h.
- (v) This group is simple:

$$SL_7(\mathbb{Z}/120\mathbb{Z}) := \{A \in (\mathbb{Z}/120\mathbb{Z})^{7\times 7} : \det A = \bar{1}\}.$$

- (vi) This group is simple:

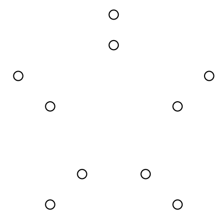
$$SL_7(\mathbb{Z}/121\mathbb{Z}) := \{A \in (\mathbb{Z}/121\mathbb{Z})^{7\times 7} : \det A = \bar{1}\}.$$

4. (10 points) Let let H be a normal subgroup of a finite G, and assume that $|H| = p$ is the smallest prime number dividing $|G|$. Show that $H \leq Z(G)$, the center of G.
(Hint: Consider the action of G on H via conjugation, that is, g sends h to ghg^{-1}. Also note that the identity e in H is fixed under this action by every g in G.)

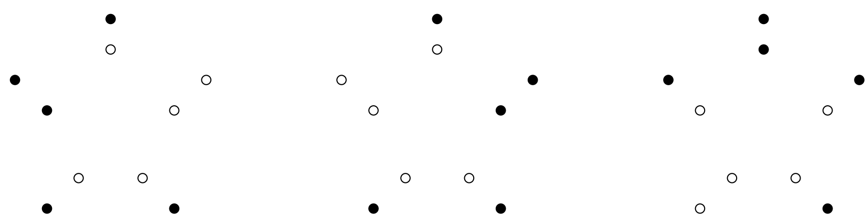
5. (15 points total) Let G be a finite group G acting on a finite set A.
(a) (5 points) Count in two ways the cardinality $|\{(g,a) \in G \times A : g(a) = a\}|$ to prove

$$|G| \cdot |\{G\text{-orbits } \mathcal{O} \text{ on } A\}| = \sum_{g \in G} |\{a \in A : g(a) = a\}|.$$

(b) (10 points) Use (a) to compute the number of orbits of the dihedral group D_{10} of cardinality 10 on the set A of cardinality $|A| = k^{10}$ consisting of all colorings with k colors of these dihedrally symmetric points:



For example with $k = 2$ colors black and white, among the 2^{10} colorings, these three lie in the same orbit



since the first two differ by reflecting across a vertical line, and the last two differ by $\frac{2\pi}{5}$ rotation.
(Hint: your answer should end up being a polynomial function of k.)

6. (15 points total) Let G be a group.

(a) (10 points) For $H \leq G$ a subgroup with finite index $n = [G : H]$, show that H contains a subgroup N which is normal in G, that is, $N \triangleleft G$, and has index $[G : N]$ dividing $n!$.
(Hint: let G act on G/H by left-translation, that is, g sends the coset aH to the coset gaH)

(b) (5 points) For subgroups H_1, H_2 of G, both of finite index in G, show $H_1 \cap H_2$ also has finite index in G.