Math 8201 Graduate abstract algebra- Fall 2019, Vic Reiner
Midterm exam 1- Due Wednesday October 16, in class

Instructions: This is an open book, library, notes, web, take-home exam, but you are *not* to collaborate. The instructor is the only human source you are allowed to consult. Indicate outside sources used.

1. (36 points total; 6 points each part)
True or false? That is, prove or disprove:

(i) If a finite group G has a quotient group G/N that contains an element of order n, then G itself contains an element of order n.

(ii) If $G/Z(G)$ is abelian, then G is abelian.

(iii) There is a group isomorphism $\mathbb{R}^\times \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{R}^+$.

(iv) For g, h in a finite group, the order of the product gh divides the product of the orders of g and h.

(v) The group $SL_{19}(\mathbb{Z}/48\mathbb{Z})$ is simple, where $SL_n(R) := \{A \in R^{n\times n} : \det A = 1\}$ for (commutative) R.

(vi) The group $SL_{19}(\mathbb{Z}/49\mathbb{Z})$ is simple.

2. (9 points) Fix a prime p. Prove that a finite group is a p-group if and only if all of its composition factors are isomorphic to $\mathbb{Z}/p\mathbb{Z}$.

3. (10 points total; 5 points each part)
(i) How many finite abelian groups are there, up to isomorphism, with cardinality 22000?
(ii) How many of those groups contain an element of order 100?

4. (30 points; 5 points each part, and feel free to assume the assertions in any previous part when doing the next part, even if you didn't complete them).

For two positive integers d, n, let $G_{d,n}$ be the set of all matrices in $\mathbb{C}^{n\times n}$ having exactly one nonzero entry in each row and each column, with all nonzero entries being d^{th} roots-of-unity in $\mathbb{C}$, that is, powers of $\zeta_d := e^{\frac{2\pi i}{d}}$. As examples, $G_{1,n} = S_n$ the set of $n \times n$ permutation matrices, and here is a matrix in $G_{6,5}$:

$$g = \begin{bmatrix} 0 & 0 & 0 & \zeta_6 & 0 \\ 0 & 0 & 0 & 0 & \zeta_6^2 \\ \zeta_6^3 & 0 & 0 & 0 & 0 \\ 0 & 0 & \zeta_6^3 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

(i) Prove that $G_{d,n}$ is a subgroup of $GL_n(\mathbb{C})$.

(ii) Compute the cardinality $|G_{d,n}|$.

(iii) Prove $G_{d,n} = \langle s_0, s_1, \ldots, s_{n-1} \rangle$, where s_i for $i = 1, 2, \ldots$ is the permutation matrix corresponding to the transposition $(i, i+1)$, while s_0 is the diagonal matrix with diagonal entries $(\zeta_d, 1, 1, \ldots, 1)$:

$$s_0 = \begin{bmatrix} \zeta_d & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

Hint: you may assume a fact asserted in lecture, that $S_n = \langle s_1, s_2, \ldots, s_{n-1} \rangle$.

(iv) Prove that the map $\chi : G_{d,n} \to \mathbb{C}^\times$ sending a matrix g in $G_{d,n}$ to the product of its nonzero entries is a group homomorphism.

(v) Prove that there are exactly $2d$ distinct group homomorphisms $G_{d,n} \longrightarrow \mathbb{C}^\times$, namely the following maps $\{\chi^e \det^\epsilon\}$ indexed $e = 0, 1, \ldots, d-1$ and $\epsilon = 0, 1$, and no others:

$$\begin{array}{rcl} \chi^e \det^\epsilon : G_{d,n} & \longrightarrow & \mathbb{C}^\times \\ g & \longmapsto & \chi(g)^e \det(g)^\epsilon. \end{array}$$

(vi) Fix a divisor c of d, and let $N \subseteq G_{d,n}$ be the subset of matrices for which the product of their nonzero entries is a c^{th} root of unity. Prove that N is a normal subgroup of $G_{d,n}$ and identify the quotient group $G_{d,n}/N$ up to isomorphism (as some much simpler group, already discussed in the book, in lecture, etc).

5. (15 points) Given two *simple* groups G_1, G_2, and a normal subgroup $N \trianglelefteq G_1 \times G_2$, show that either

- $N = \{(e, e)\}$, that is $|N| = 1$, or
- $N = G_1 \times G_2$, or
- N is *isomorphic* to at least one of G_1 or G_2.