Math 8201 Graduate abstract algebra- Fall 2001, Vic Reiner Final exam - Due Friday December 21, by 5PM in my mailbox- NO EXCEPTIONS

Instructions: This is an open book, open library, open notes, take-home exam, but you are *not* to collaborate. The instructor is the only human source you are allowed to consult. Each of the 8 problems is worth approximately the same number of points. **Please note:** if you cannot do one part of one problem, you are still free to assume its correctness in doing other parts of the exam.

1. Let $\mathcal{F} = \{\phi_\alpha\}$ be any family of pairwise commuting operators on a finite dimensional $\mathbb{C}$-vector space.
(a) Show that they can be *simultaneously triangularized*, that is, there exists a basis $(v_1, v_2, \ldots, v_n)$ for V with the property that for every α and every i one has $\phi_\alpha(v_i)$ lying in the span of $v_1, v_2, \ldots, v_i$.
(b) Show that if each ϕ_α is diagonalizable, then they can be *simultaneously diagonalized*, that is, there exists a basis $(v_1, v_2, \ldots, v_n)$ for V such that v_i is an eigenvector for every ϕ_α.

2. Let V be an $\mathbb{F}$-vector space of (finite) dimension n. Define the *trace* $\text{trace}(\phi)$ of a linear operator $\phi : V \to V$ as follows: let $A = (a_{ij})$ be the matrix representing ϕ with respect to any choice of a basis $(v_1, \ldots, v_n)$ for V, i.e. $\phi(v_j) = \sum_{i=1}^n a_{ij} v_i$, and then define

$$\text{trace}(\phi) := \text{trace}(A) := \sum_{i=1}^n a_{ii}.$$

(a) Prove that $\text{trace}(\phi)$ is well-defined, that is, independent of the choice of basis $(v_1, \ldots, v_n)$.
(b) Prove that the adjoint map $\phi^* : V^* \to V^*$ defined on functionals $f \in V^*$ by $\phi^*(f) = f \circ \phi$ satisfies

$$\text{trace}(\phi^*) = \text{trace}(\phi).$$

(c) Let W be a ϕ-invariant subspace, that is, $\phi(W) \subset W$. Then we have seen (in homework) that this induces operators $\phi|_W : W \to W$ and $\phi_{V/W} : V/W \to V/W$. Prove that in this situation

$$\text{trace}(\phi) = \text{trace}(\phi|_W) + \text{trace}(\phi_{V/W}).$$

3. Let $\phi : V \to V$ be a linear operator on an $\mathbb{F}$-vector space V of (finite) dimension n, with $\mathbb{F}$ algebraically closed.
(a) Show that the coefficients in its characteristic polynomial have the following interpretation:

$$\det(t \cdot 1_V - \phi) = \sum_{k=0}^{n} (-1)^k \operatorname{trace}(\wedge^k \phi : \wedge^k V \to \wedge^k V) \cdot t^{n-k}.$$

(Hint: Use the fact that $\mathbb{F}$ is algebraically closed to triangularize ϕ, then prove the result when ϕ is triangularized. Note also that one can remove the assumption that $\mathbb{F}$ is algebraically closed by passing to its algebraic closure, but we won't worry about this now.)
(b) Let $\phi : V \to W$ and $\psi : W \to V$ be linear maps between vector spaces V, W of dimensions n, m respectively, with $m \leq n$. Deduce from part (a) that

$$\det(t \cdot 1_V - \psi\phi) = t^{n-m} \det(t \cdot 1_W - \phi\psi).$$

4. Note that in this problem, since the later parts generalize the earlier parts, you should only give a solution to one of them. You get $\frac{1}{2}$ the total number of points possible for doing (a), $\frac{3}{4}$ for doing (b), and all the points for doing (c), but do not get more points for doing more than one part.
(a) Let

$$0 \to C_r \xrightarrow{\partial_r} C_{r-1} \to \cdots \to C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \to 0$$

be an exact sequence of finite dimensional vector spaces C_i and linear maps, that is, $\ker(\partial_i) = \operatorname{im}(\partial_{i+1})$. Prove that

$$\sum_{i \geq 0} (-1)^i \dim_{\mathbb{F}} C_i = 0.$$

(b) Prove the following more generally. Given any finite-dimensional *chain complex*, that is a sequence of finite-dimensional vector spaces and linear maps

$$0 \to C_r \xrightarrow{\partial_r} C_{r-1} \to \cdots \to C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \to 0$$

in which $\partial_i \circ \partial_{i+1} = 0$, define the *homology spaces* to be the quotient spaces

$$H_i := \ker \partial_i / \operatorname{im} \partial_{i+1}.$$

Prove that
$$\sum_{i\geq 0}(-1)^i \dim_{\mathbb{F}} H_i = \sum_{i\geq 0}(-1)^i \dim_{\mathbb{F}} C_i.$$
(Note: this does generalize (a), since exactness is the same as $H_i = 0$ for all i.)

(c) Prove the following more generally. Say that a sequence of linear operators $\phi_i : C_i \to C_i$ forms a *morphism* of chain complexes if
$$\partial_i \circ \phi_i = \phi_{i-1} \circ \partial_i$$
for each i. Show that for any morphism of chain complexes,
$$\phi_i(\ker \partial_i) \subset \ker \partial_i$$
$$\phi_i(\mathrm{im}\partial_{i+1}) \subset \mathrm{im}\partial_{i+1}$$
implying that there is induced map $\phi_i : H_i \to H_i$, and one has
$$\sum_{i\geq 0}(-1)^i \mathrm{trace}(\phi_i : H_i \to H_i)$$
$$= \sum_{i\geq 0}(-1)^i \mathrm{trace}(\phi_i : C_i \to C_i)$$
(Hints for parts (b) and (c): There are short exact sequences
$$0 \to \mathrm{im}\partial_{i+1} \to \ker \partial_i \to H_i \to 0$$
$$0 \to \ker \partial_i \to C_i \to \mathrm{im}\partial_i \to 0$$
and Problem 2(c) is relevant for part (c). Note also that the answer for this part does generalize (b), by taking $\phi_i = 1_{C_i}$ for each i.)

5. Let G be a finite group of order $2k$ with k odd. Show that there exists a normal subgroup of index 2 in G. (Hint: consider the action of G on itself by left-translation, and use the fact that G contains at least one element of order 2.)

6. Show that the alternating group A_4 has $Aut(A_4)$ strictly larger than $Inn(A_4)$. (Hint: consider the action of S_4 on A_4 by conjugation.)

7. Let G_1, G_2 be simple groups, and $N \lhd G_1 \times G_2$. Show that either $N = 1, N = G_1 \times G_2$, or N is isomorphic to G_1 or G_2.

8. Let V be an $\mathbb{F}$-vector space of (finite) dimension n, and V^* its dual space.
(a) Show that the map

$$\begin{aligned} V \otimes V^* &\overset{\alpha}{\to} Hom(V,V) \\ v \otimes f &\mapsto (\phi : w \mapsto f(w) \cdot v) \end{aligned}$$

is well-defined, and an isomorphism of vector spaces.
(b) Let β be any isomorphism $Hom(V,V) \cong Mat_{n\times n}(\mathbb{F})$ that comes from choosing a basis for V. Show that under the composite isomorphism $\beta \circ \alpha : V \otimes V^* \to Mat_{n\times n}(\mathbb{F})$, the subset of decomposable tensors $v \otimes f$ bijects with the subset of matrices of rank at most 1.