Math 8201 Graduate abstract algebra- Fall 2001, Vic Reiner
Linear algebra homework problems for HW 6
(mostly taken from Hoffman and Kunze's text, *Linear algebra*)

1. Show that

$$\langle A, B\rangle := \mathrm{trace}(A(B^*))$$

defines an inner product on the $\mathbb{C}$-vector space $V = M_{n\times n}(\mathbb{C})$.

2. Prove a fact (used in lecture) that the Vandermonde matrix $A = (a_{ij})_{i,j=1,\ldots,n}$ defined by $a_{ij} = \lambda_j^{i-1}$ has determinant

$$\prod_{1\leq i<j\leq n} (\lambda_j - \lambda_i).$$

(Hint: There are several proofs, e.g. by induction on n using row operations, or by using the fact that the determinant vanishes when $\lambda_i = \lambda_j$).

3. Show that the product of two self-adjoint operators is self-adjoint if and only if the two operators commute.

4. Let V be the vector space of polynomials of degree at most 3 with $\mathbb{C}$ coefficients, and the inner product

$$\langle f, g\rangle := \int_0^1 f(t)\overline{g(t)}dt.$$

Let D be the differentation operator. Find D^*.

5. Let V be a finite-dimensional inner product space, and $E : V \to V$ and idempotent operator, i.e. $E^2 = E$. Prove that E is self-adjoint ($E^* = E$) if and only if E is normal ($E^*\,E = E\,E^*$).

6. Working in $M_{2\times 2}(\mathbb{C})$, find an explicit unitary matrix U and diagonal matrix D such that the rotation matrix

$$A = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

has $U^*AU = D$.

7. Prove that an operator $\phi : V \to V$ on an inner product space V is normal if and only if $\phi = \phi_1 + i\phi_2$ where ϕ_1, ϕ_2 are self-adjoint operators that commute.